Baryons in 1/N expansion

E. Witten, Nucl. Phys. B160 (1979) 57-115
arXiv: 9310369 [hep-ph]

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Contents

Introduction : large-N gauge theory

From now on, we consider a gauge theory with large N.

→ Why large N ? [1974, 't Hooft]

1) Gives non-perturbative results beyond the ordinary coupling expansion.

2) Agrees with the $N = 3$ properties

3) Corresponds to perturbative expansion in string theory

→ Consider $1/N$ expansion with fixed $\lambda \equiv g^2 N$ ('t Hooft coupling)

$a = \frac{g^2}{16\pi^2}$

(※ To avoid divergences in β-fn. : $\mu \frac{\partial a}{\partial \mu} = -\left(\frac{11}{3} N a^2 + \frac{31}{3} N^2 a^3 + \cdots \right)$)

Introduction: large-N gauge theory

SU(N) gauge theory

$$S[\psi, \bar{\psi}, A_\mu] = \int d^4x \left[-\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}^2 + \bar{\psi}(i\not{D} - m)\psi \right]$$

【Components】

- Fermions ψ^i $(i = 1, 2, \cdots, N)$
- Anti-fermions $\bar{\psi}_j$ $(j = 1, 2, \cdots, N)$
- Gauge bosons $A_\mu{}^i{}_j$ $(i, j = 1, 2, \cdots, N)$

· "Color" degrees of freedom

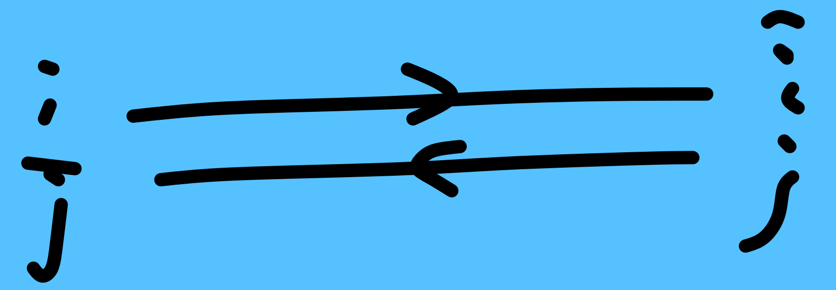
$$\psi^i, \bar{\psi}_j \sim \mathcal{O}(\underline{\underline{N}}) \quad , \quad A_\mu{}^i{}_j \sim \mathcal{O}(\underline{\underline{N^2}})$$

$$\left(※\ \dim SU(N) = N^2 - 1 \simeq N^2 \right)$$

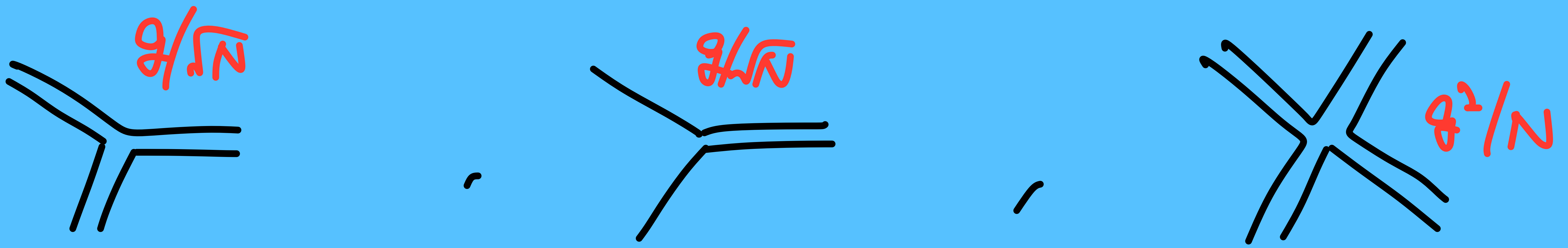
Introduction : large-N gauge theory

To show the "color" structures explicitly, we introduce **Double-line Notation.**

Propagator

- Fermions $\quad \psi^i \quad : \quad i \longrightarrow \bar{i}$
- Anti-fermions $\quad \bar{\psi}_j \quad : \quad j \longleftarrow \bar{j}$
- Gauge bosons $\quad A_\mu{}^i{}_j \quad :$ 

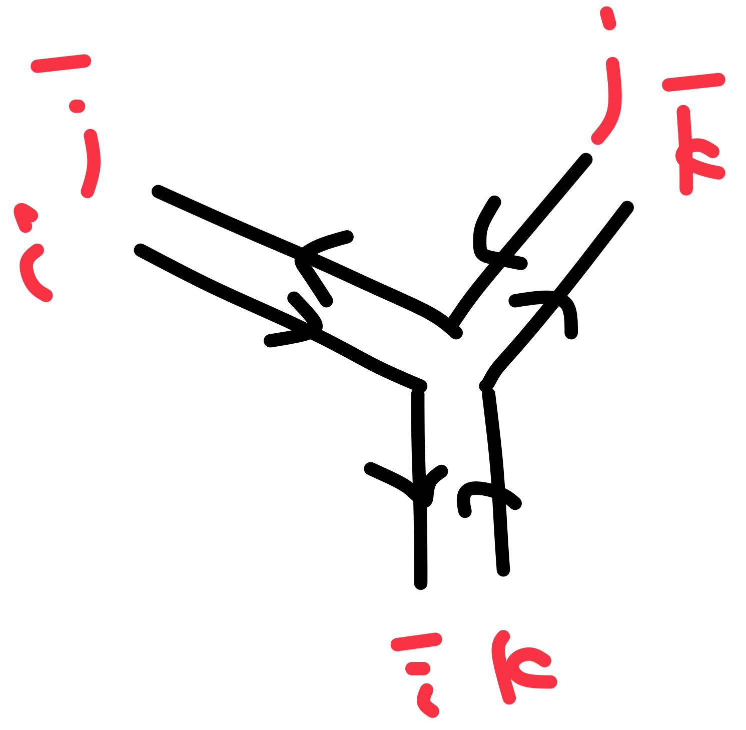
Vertex



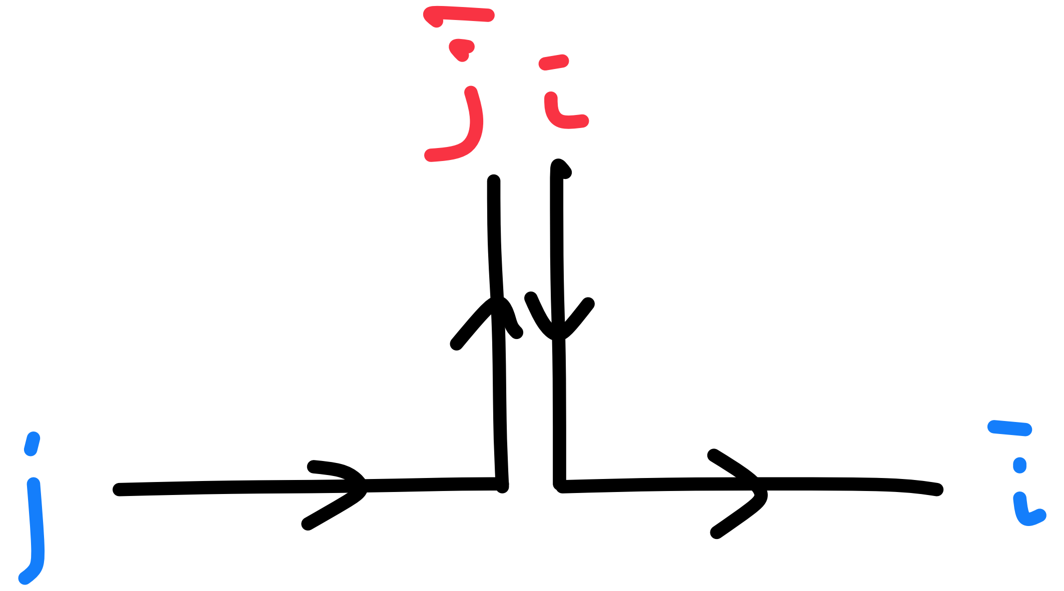
Introduction: large-N gauge theory

Examples

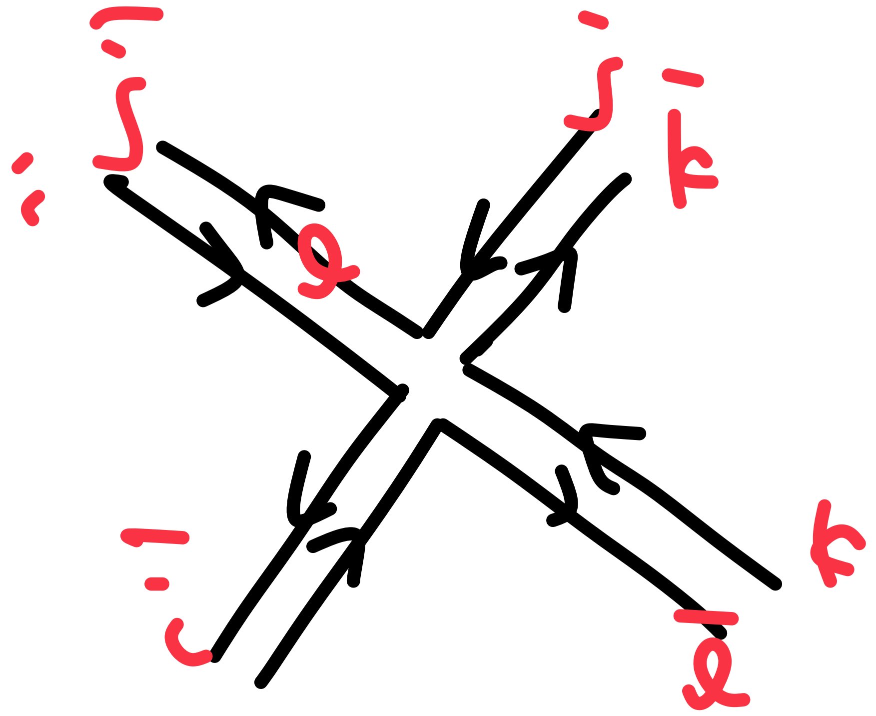
- $\mathrm{Tr}(A_\mu A_\nu \partial_\mu A_\nu) = A_\mu{}^i{}_j \, A_\nu{}^j{}_k \, \partial_\mu A_\nu{}^k{}_i =$



- $\bar{q} \not{A} q = \bar{q}_i \gamma^\mu q^j \, A_\mu{}^i{}_j =$



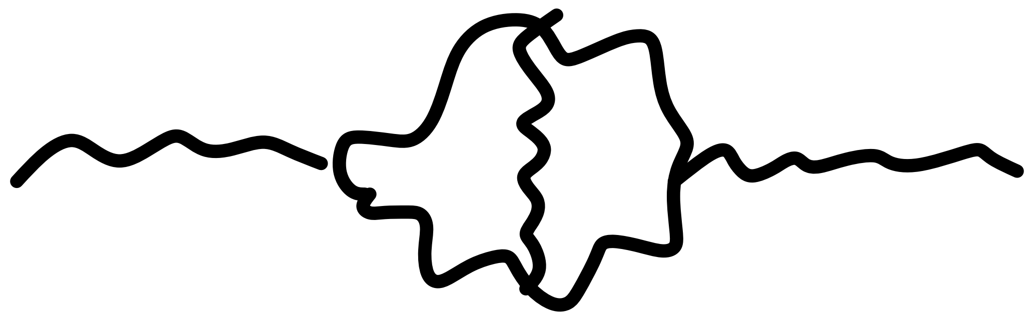
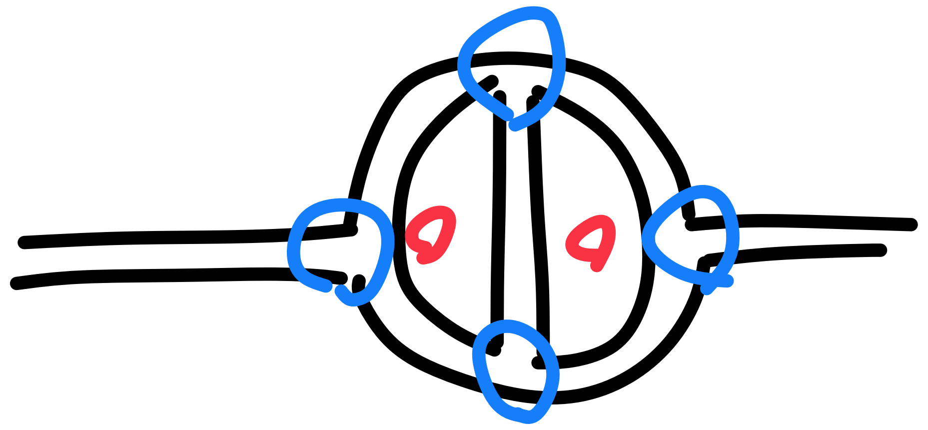
- $\mathrm{Tr}(A_\mu A_\nu A_\mu A_\nu) = A_\mu{}^i{}_j \, A_\nu{}^j{}_k \, A_\mu{}^k{}_l \, A_\nu{}^l{}_i =$



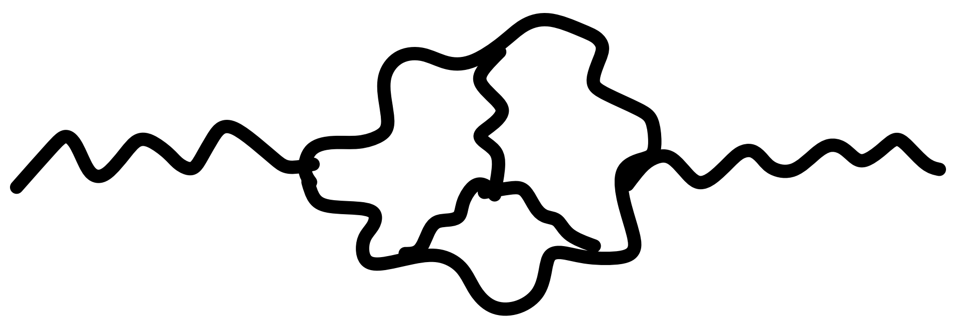
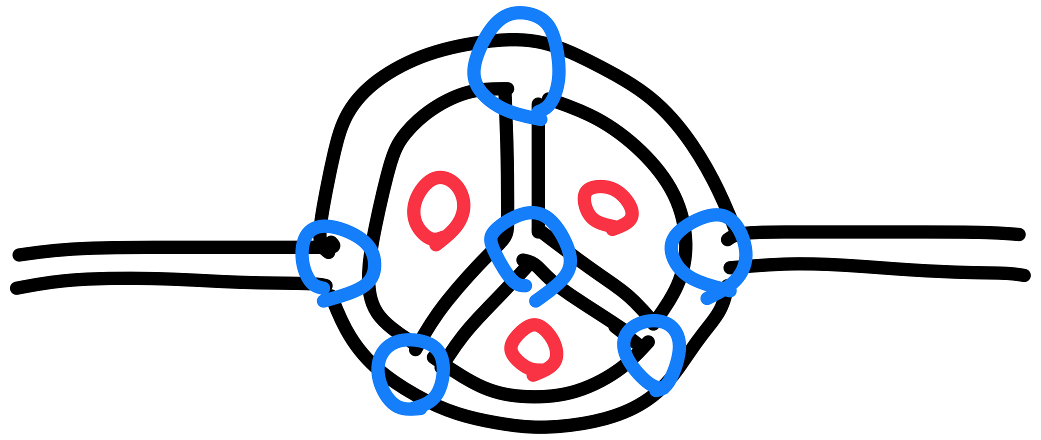
Introduction: large-N gauge theory

Using the notation, we can determine which diagrams will survive in the limit.

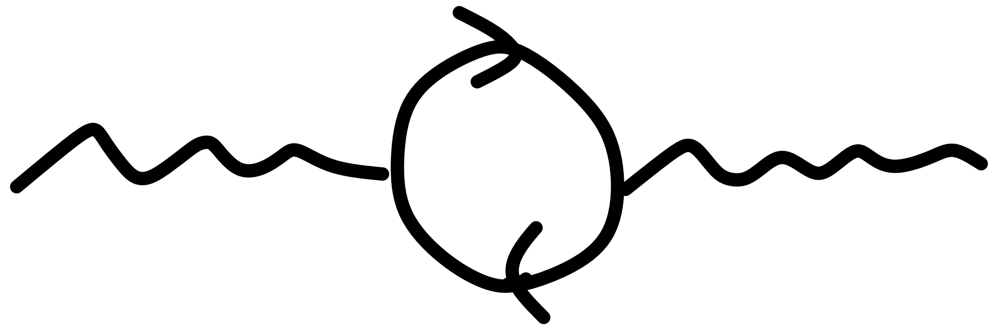
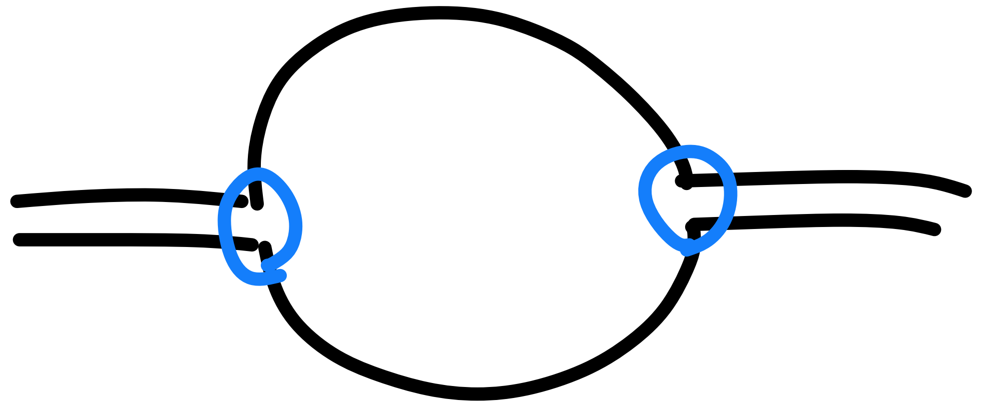
Examples

 =  $\sim \mathcal{O}\left((1/\sqrt{N})^4 \cdot N^2\right) = \mathcal{O}(1)$

$\longrightarrow$ survive

 =  $\sim \mathcal{O}\left((1/\sqrt{N})^6 \cdot N^3\right) = \mathcal{O}(1)$

$\longrightarrow$ survive

 =  $\sim \mathcal{O}\left((1/\sqrt{N})^2\right) = \mathcal{O}(1/N)$

$\longrightarrow$ vanish

Introduction : large-N gauge theory

$$\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^6 N \right) = \mathcal{O}(1/N^2)$$

$\rightarrow$ vanish

Generally, diagrams that survive the limit should ...

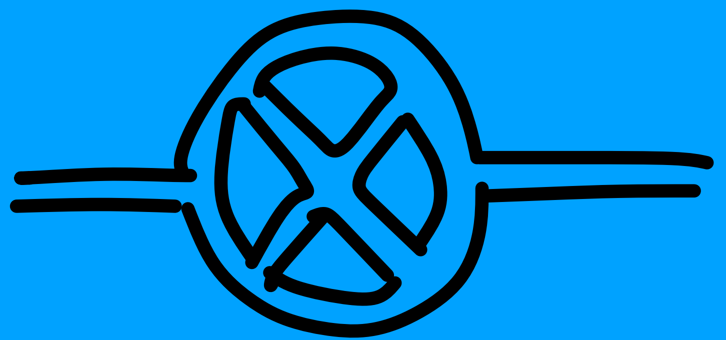
1) Not include fermion loops

- the difference in DoF : $\underset{\mathcal{O}(N^2)}{A_\mu} \longleftrightarrow \underset{\mathcal{O}(N)}{\psi}$

$\Rightarrow$ Diagrams with the loops $\sim \mathcal{O}(1/N)$

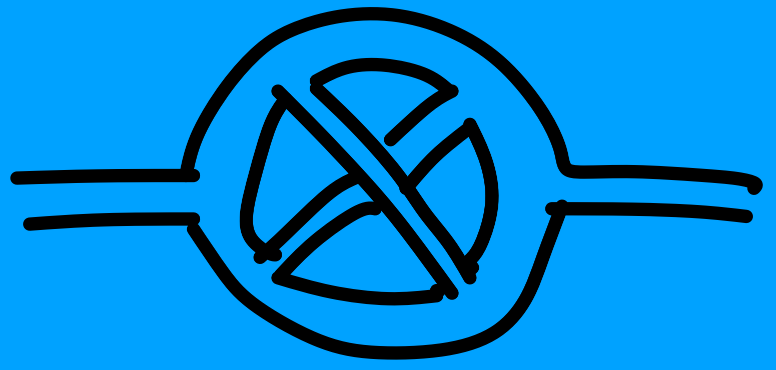
2) Be planar

$\Rightarrow$ Non-planar diagrams $\sim \mathcal{O}(1/N^2)$



planar

$\sim \mathcal{O}(1)$



non-planar

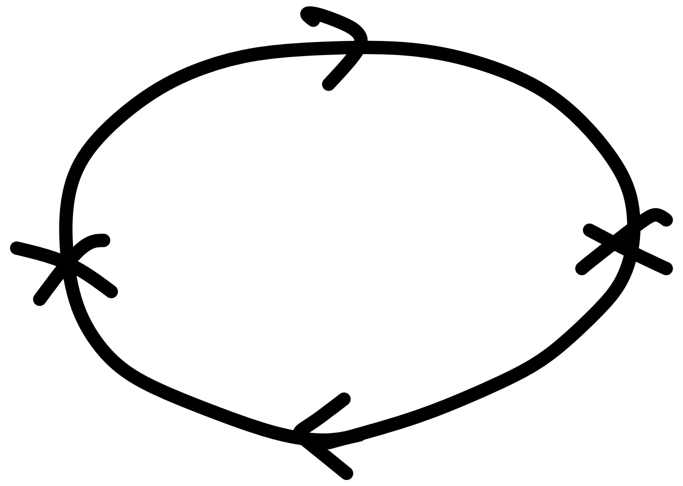
$\sim \mathcal{O}(1/N^2)$

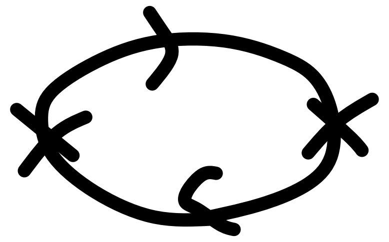
can be put on a mfd without holes

Mesons for large N

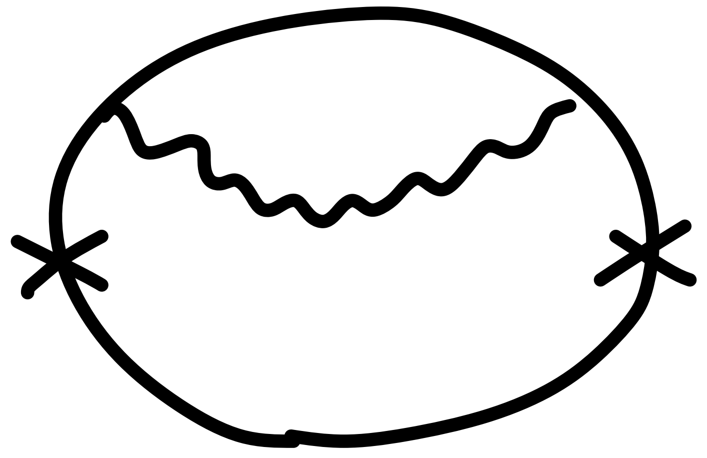
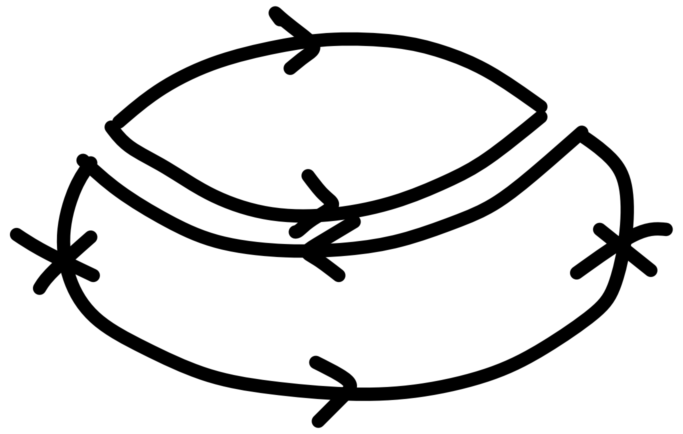
Now we consider two-pt functions of quark bilinears

($J = \bar{q}q, \ \bar{q}\gamma^\mu q$)

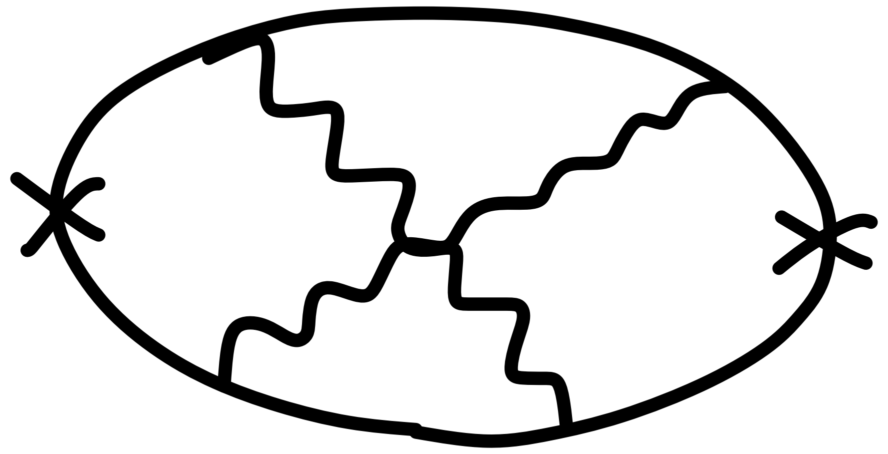
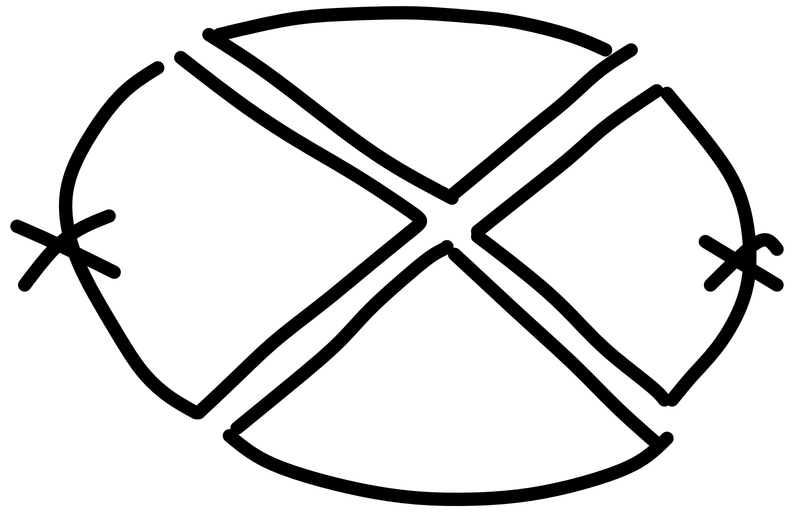
· Free  $\sim \mathcal{O}(N)$

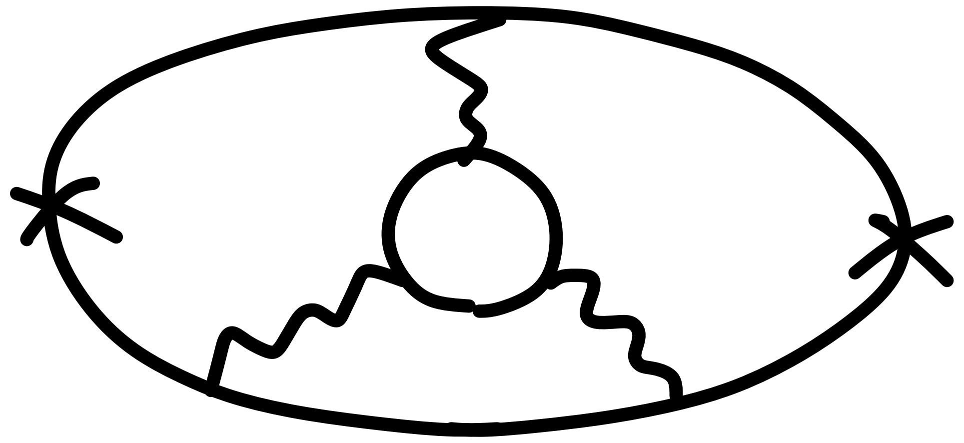
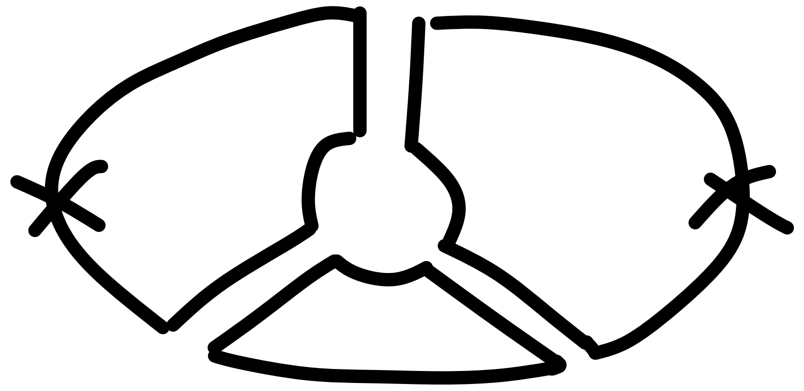
If we insert a planar propagator of the gauge boson ($\mathcal{O}(N^0)$) into , its N-dependence will remain unchanged.

Examples

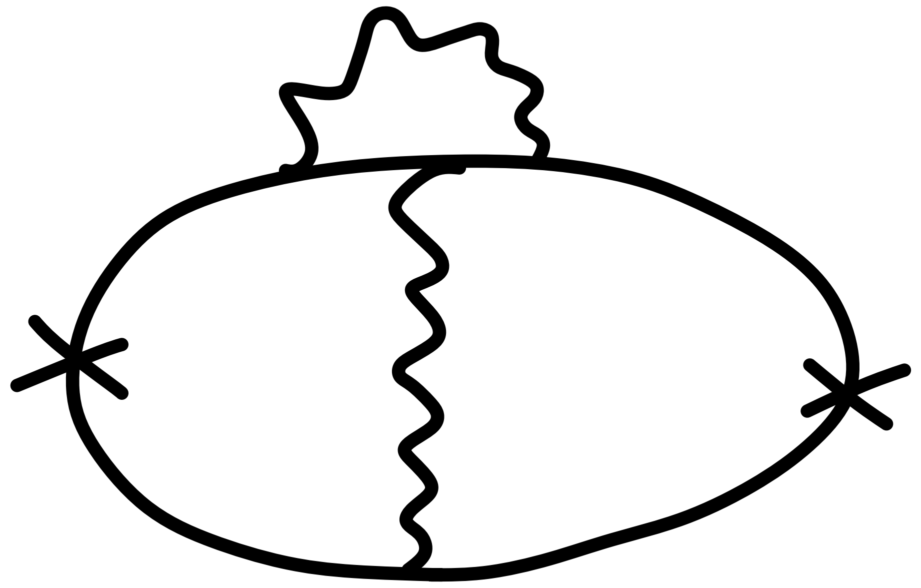
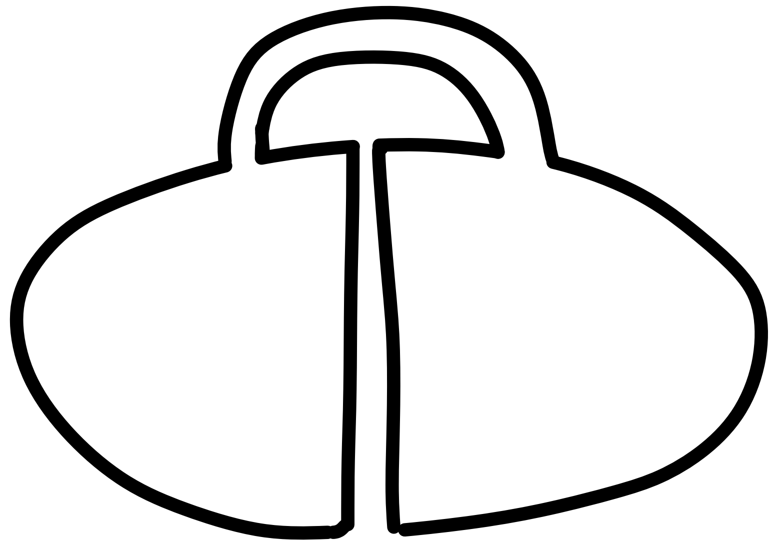
·  =  $\sim \mathcal{O}\left((1/\sqrt{N})^2 \cdot N^2\right) = \mathcal{O}(N)$

Mesons for large N

-  $=$  $\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^6 \cdot N^4\right) = \mathcal{O}(N)$

-  $=$  $\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^6 \cdot N^3\right) = \mathcal{O}(1)$

  $\rightarrow$ subdominant

-  $=$  $\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^4\right) = \mathcal{O}(1/N^2)$

  $\rightarrow$ subdominant

$\rightarrow$ Gluon line should not come to the edge of a diagram.

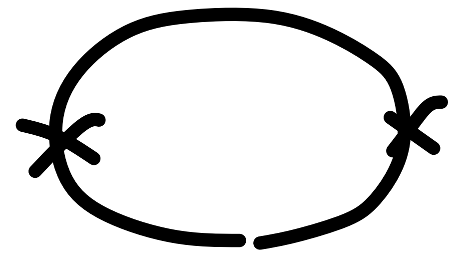
Mesons for large N

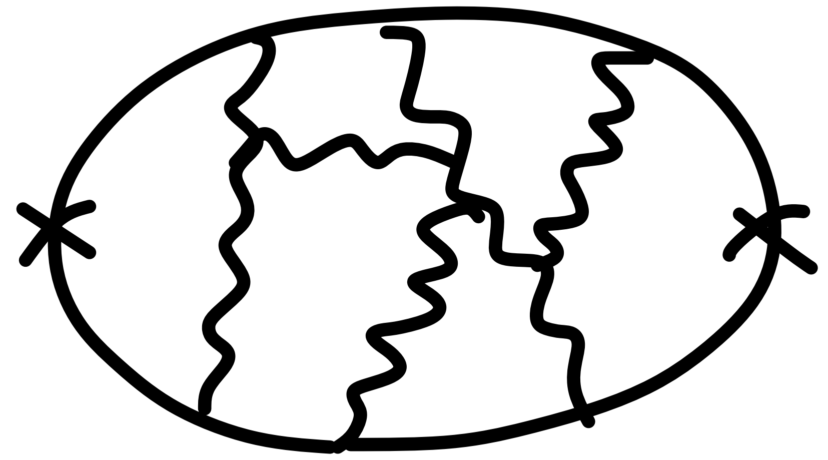
The dominant contribution to matrix elements of quark bilinears J comes from planar diagrams

- with only fermion lines at the edges
- with no fermion loops

Namely,

$\sim \mathcal{O}(N)$

$\langle JJ \rangle =$  $+ \sum \underline{\underline{(\text{others})}} + \mathcal{O}(N^0)$

a typical diagram : 

Mesons for large N

We will see the qualitative picture of mesons.

Assumption:
The confinement persists also at large N

→ What can be deduced about mesons?

created by op. $J(x)$

At the $1/N$ leading order:

1. $J(x)|0\rangle$ includes only meson's 1-particle state
2. The number of mesons is infinite.
3. Mesons are stable and non-interacting.

Mesons for large N

1. $J(x)|0\rangle$ includes only meson's 1-particle state

Namely,

$$\langle J(k) J(-k)\rangle = \langle J(k)\rangle^2 + \sum_n \langle 0|J(k)|n\rangle \frac{1}{k^2-m_n^2}\langle n|J(-k)|0\rangle$$

$$+ (\text{multi-particle state})$$

$$= \underbrace{\sum_n \frac{a_n^2}{k^2-m_n^2}}_{\text{1-particle state}} + \cancel{\int_{\sim 4m_0^2}^{\infty} dM^2\, \rho(M^2)\, \frac{1}{k^2-M^2}}$$

$\rightarrow$ This can be deduced if the theory is confining.

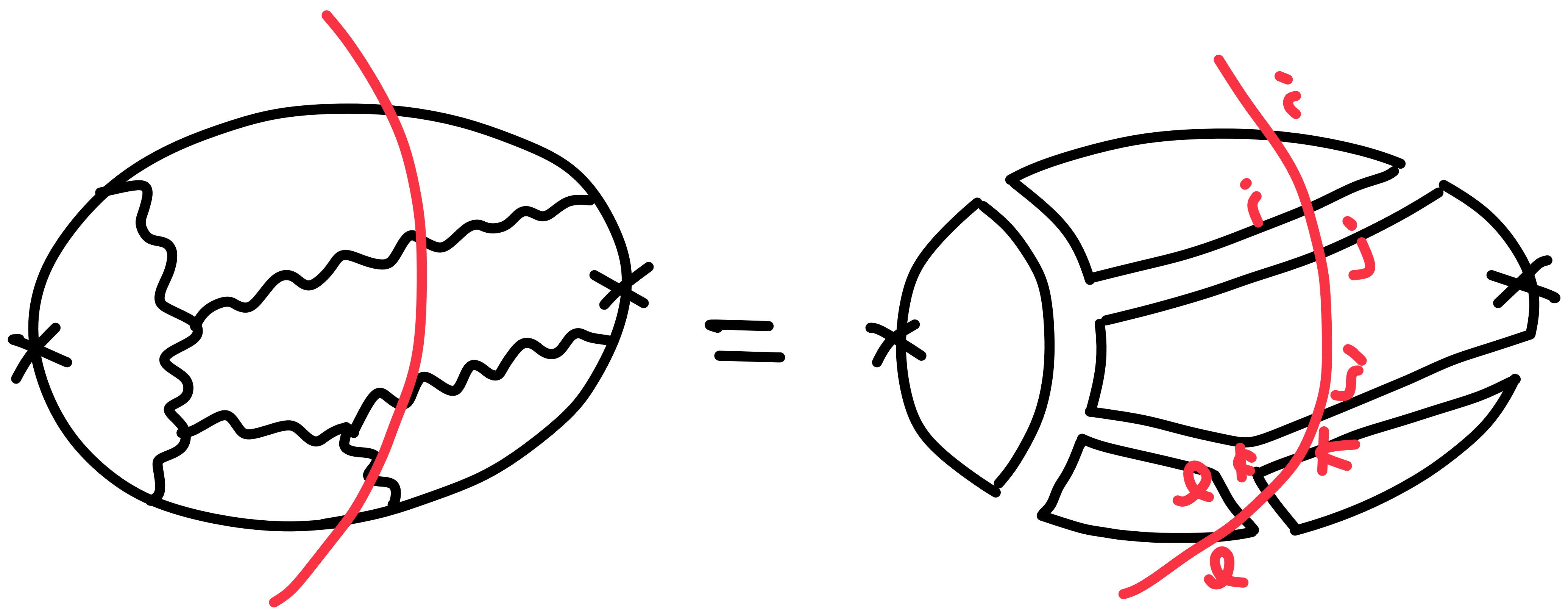
Mesons for large N

1. $J(x)|0\rangle$ includes only meson's 1-particle state

☺ Intermediate states should be color-singlet states.

- 1-particle intermediate state

$$\underbrace{\bar{q}_i A^i{}_j A^j{}_k A^k{}_l q^l}_{\text{singlet}}$$



→ planar

- Multi-particle intermediate state

$$\underbrace{\bar{q}_i A^i{}_j q^j}_{\text{singlet}} \underbrace{A^k{}_l A^l{}_k}_{\text{singlet}}$$ → corresponds to a non-planar diagram

Mesons for large N

1. $J(x)|0\rangle$ includes only meson's 1-particle state

Thus,

$$\underbrace{\langle J(k) J(-k) \rangle}_{\substack{\text{Sum of planar diagrams} \\ \sim \mathcal{O}(N)}} = \sum_n \frac{a_n^2}{k^2 - m_n^2} \quad \sim \mathcal{O}(N)$$

Since (LHS) has a smooth limit for large N, (RHS) also has a smooth limit (without any singularities).

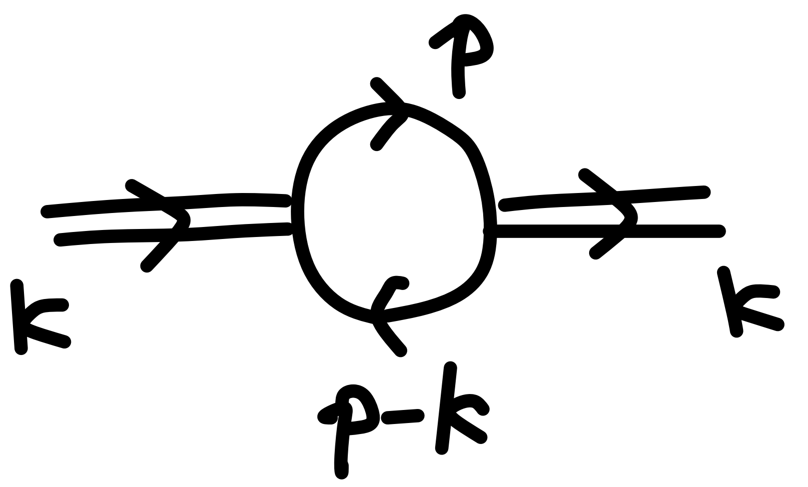
- $a_n = \langle 0 | J(k) | n \rangle \sim \mathcal{O}(\sqrt{N})$
- $m_n \sim \mathcal{O}(1)$ (independent of N)

Mesons for large N

2. The number of mesons is infinite.

In $k^2 \gg \Lambda^2$ region, purterbation theory is valid because of asymptotic freedom.

$$\langle J(k) J(-k) \rangle \sim$$



$$\underset{k^2 \gg \Lambda^2}{\sim} \mathcal{O}(k^2 \ln k^2)$$

On the other hand, if $\sum_n$ is a finite summation,

$$\langle J(k) J(-k) \rangle \sim \sum_n \frac{a_n^2}{k^2 - m_n^2} \underset{k^2 \gg \Lambda^2}{\sim} \mathcal{O}(1/k^2).$$

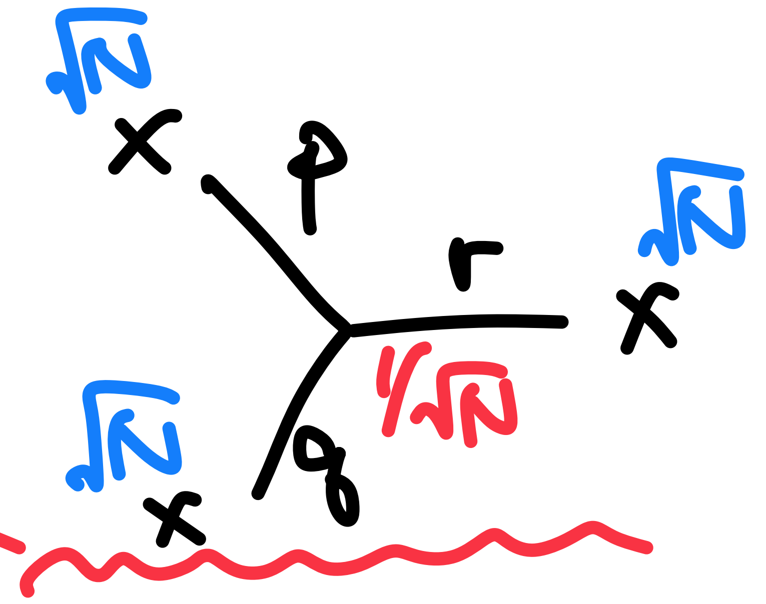
→ $\sum_n$ should be infinite sum to reproduce the logarithmic dependence.

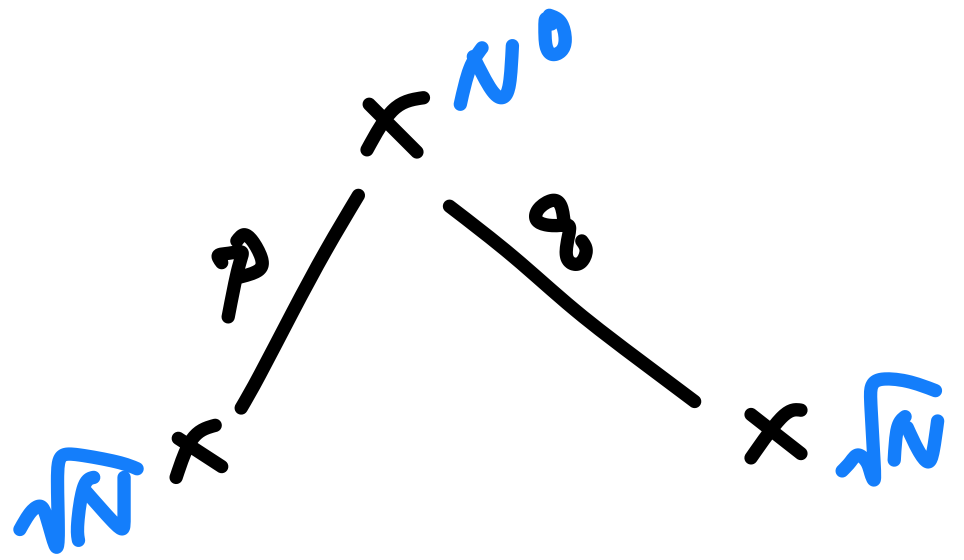
Mesons for large N

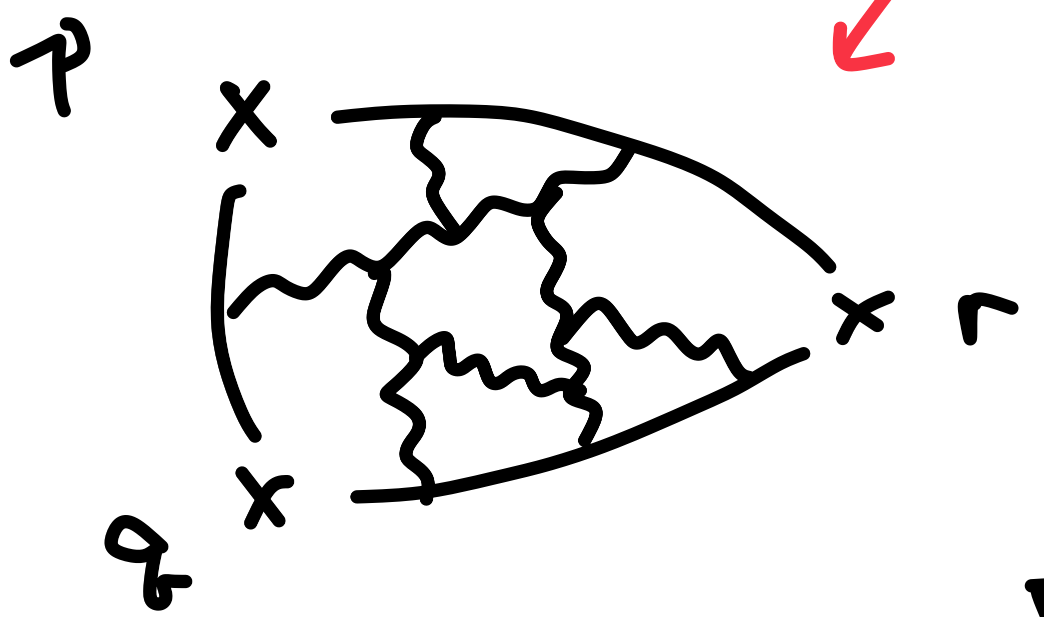
In short,

$$\langle JJ\rangle = \sum_n \underset{a_n}{\overset{\sqrt{N}}{\times}} \xrightarrow{\frac{1}{k^2-m_n^2}} \underset{a_n}{\overset{\sqrt{N}}{\times}} \sim O(N)$$

3-pt function

$\langle JJJ\rangle = \sum$ 

$+ \sum$ 



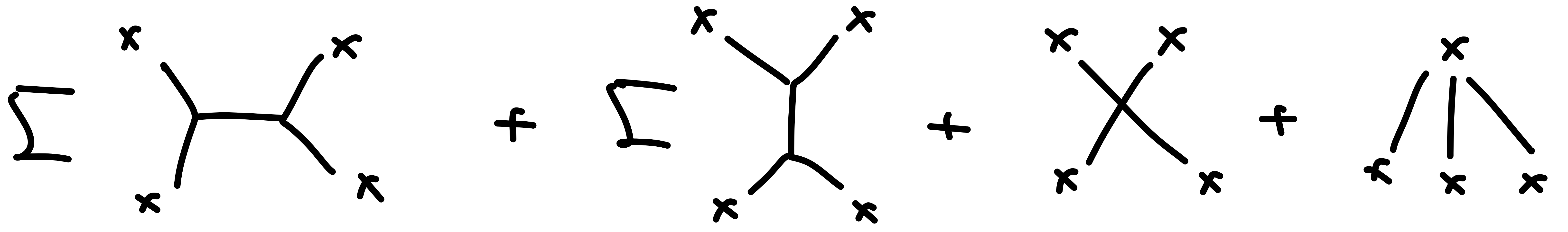
$$\frac{A}{(p^2-a^2)(q^2-b^2)(r^2-c^2)} \qquad \frac{B}{(p^2-a^2)(q^2-b^2)}$$

$$= \sum \langle 0|J|m\rangle^3\, \Gamma_{mmm} + \sum \langle 0|J|m\rangle^2 \langle 0|J|mm\rangle\, \Gamma_{mmm} \sim O(N)$$

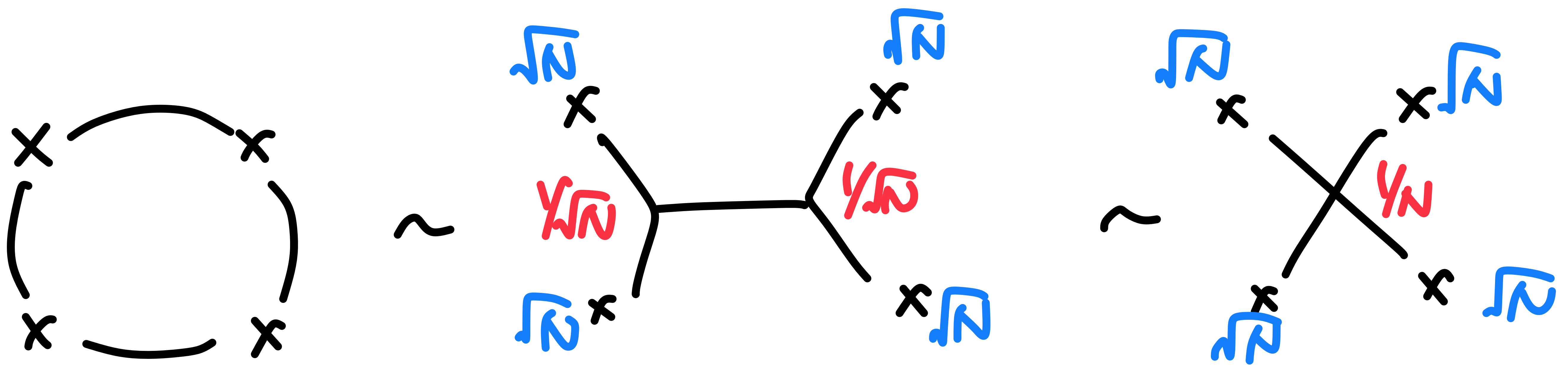
Thus, **amplitude Γ_{mmm} $(A\to BC)$ is of order $1/\sqrt{N}$** $(\to 0$ for large $N)$

Mesons for large N

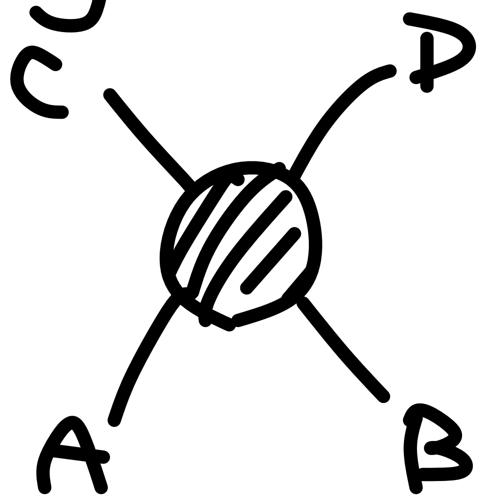
4-pt function

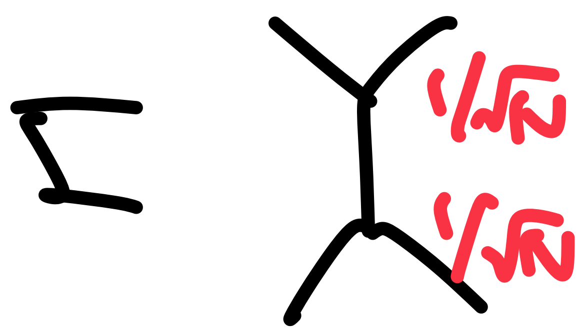
$\langle JJJJ \rangle =$  $+ \cdots$

※ Order counting

$O(N) \sim$ 

※ 2-body scattering $(AB \to CD)$



$=$ 

$+ \cdots \sim O(1/N)$ $(\to 0$ for large $N)$

Mesons for large N

Short Summary

for large N,

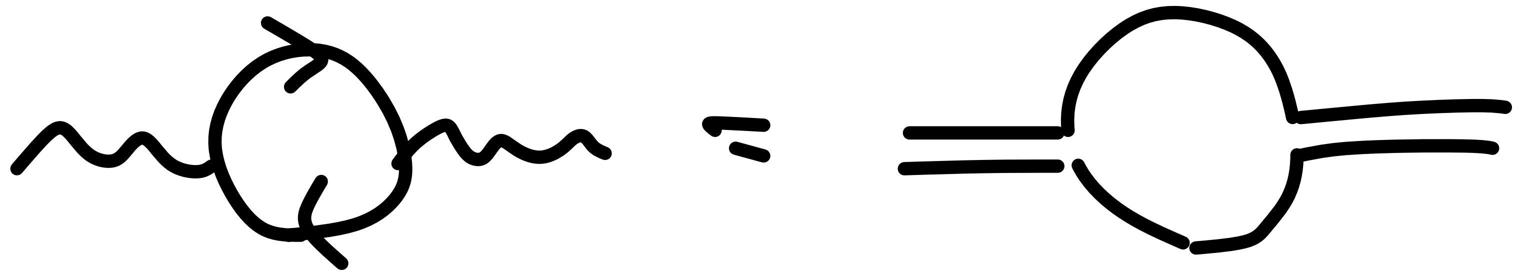
- mesons are stable and non-interacting
- leading contribution comes from tree diagrams with hadron exchanges

→ Next, we try to deduce some phenomenological properties of mesons from the analysis for large N.

Mesons for large N

【 Phenomenological properties 】

(i) Sea quarks $\bar{q}q$ is suppressed in a meson.

 $\sim \mathcal{O}(1/N)$

$\rightarrow$ Mesons are purely $\bar{q}q$ states without any sea quarks.

Exotics (e.g. $\bar{q}\bar{q}qq$) do not exist.

$\bar{q}q$ and $\bar{q}q$ are non-interacting

(Amplitudes involving two mesons vanish for large N)

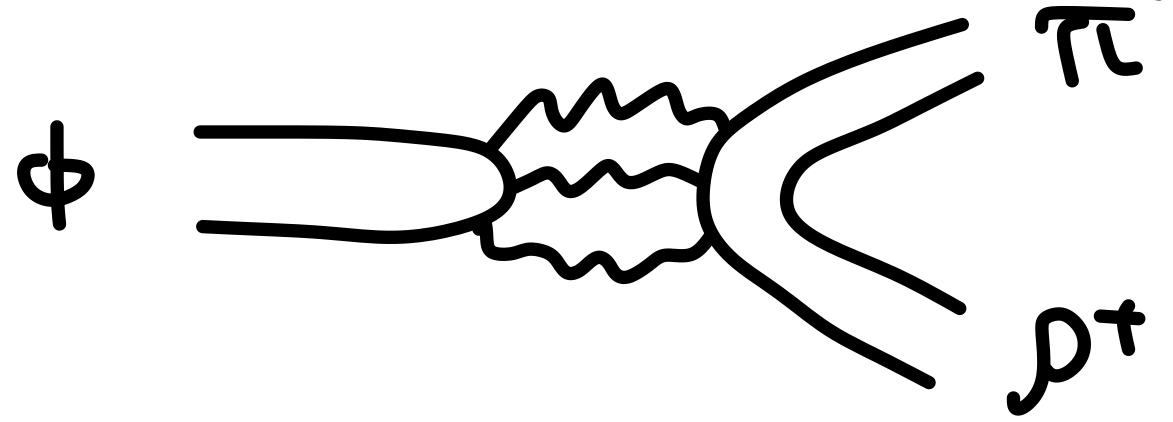
Mesons for large N

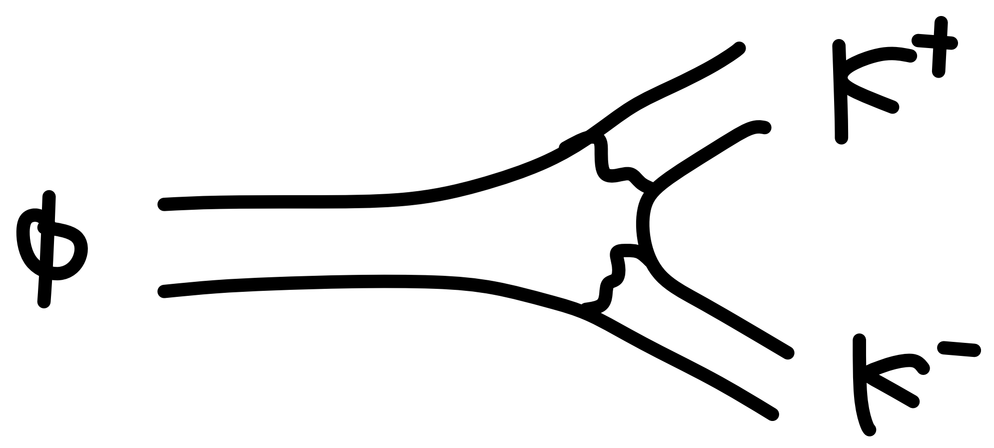
Phenomenological properties

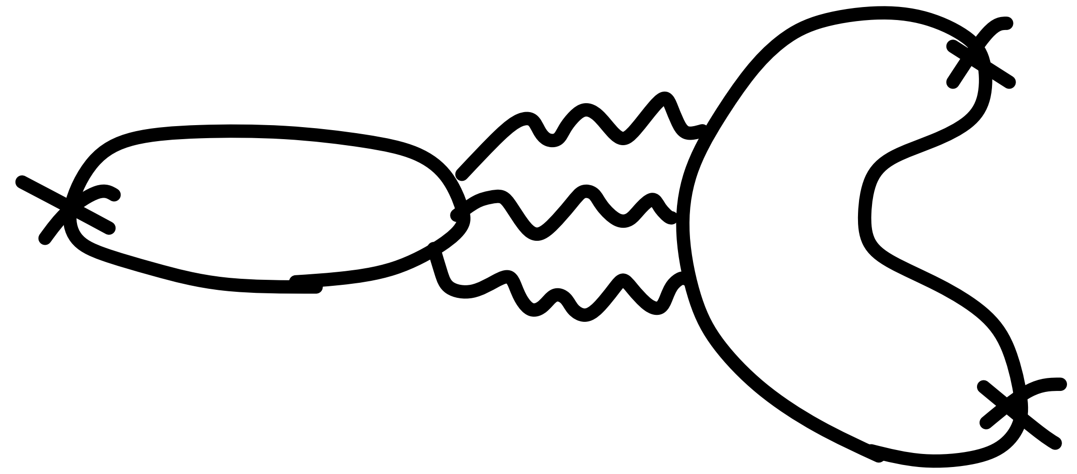
(ii) OZI rule is justified for large N

cf. A decay channel with disconnected quark lines is strongly suppressed.

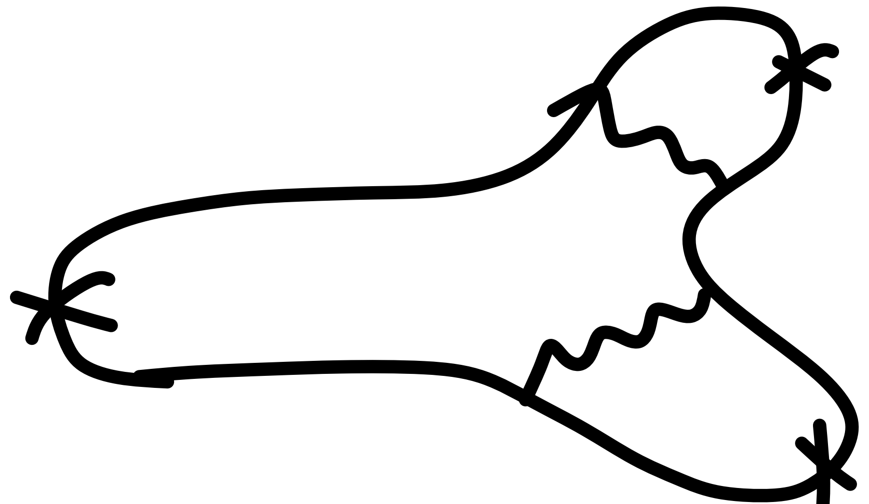
e.g. $\phi \to \pi^- + \rho^+ \ll \phi \to K^+ + K^-$







$\sim \dfrac{1}{N}$

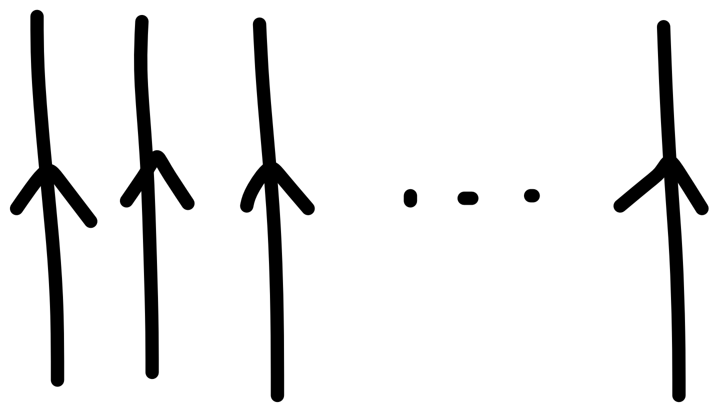


$\sim O(1)$ $\qquad\qquad$ $\sim O(N)$

Baryons for large N

The treatment of baryons differs from that of mesons.

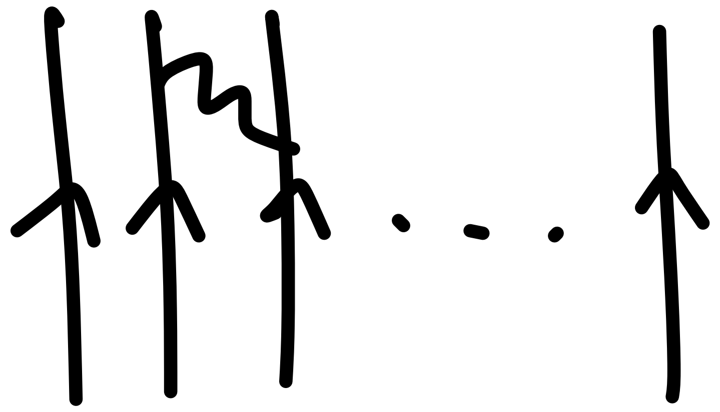
☆ **Baryon propagator**



← The problem is that diagrams depend on N.

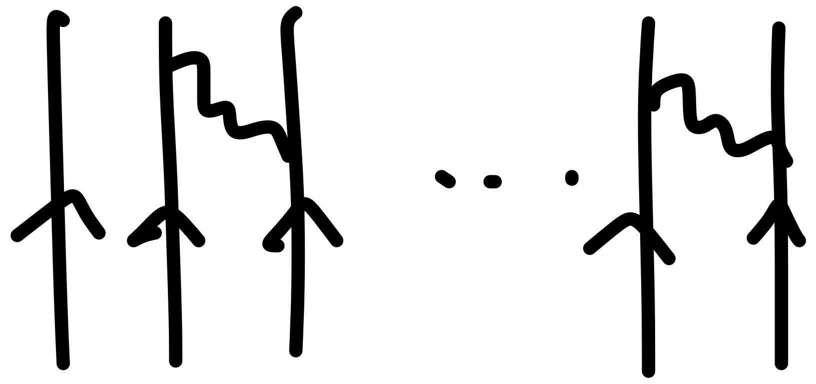
∘ Correction to baryon propagators (N-quark state):

leading



$\ll$

next-leading



$\ll \cdots$

$$\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^2 \cdot \underbrace{N^2}_{\text{combination factor}}\right) \sim \mathcal{O}(N)$$

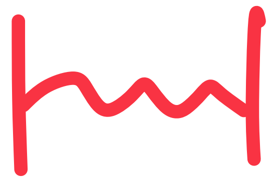
$$\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^4 \cdot \underbrace{N^4}_{\text{Combination factor}}\right) \sim \mathcal{O}(N^2)$$

Baryons for large N

→ Diagrams do not work well because N-dependences would diverge.

☆ Hamiltonian formalism

two-body interaction



$\sim \mathcal{O}((1/\sqrt{N})^2)$

$= \mathcal{O}(1/N)$

Baryon mass :

$$M_B = N\underbrace{M}_{\text{quark mass}} + N\underbrace{T}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}N^2}_{\text{combinatoric}}\left(\frac{1}{N}\cdot V\right)$$

$$= N\left(M + T + \frac{1}{2}V\right) \sim \mathcal{O}(N)$$

Baryons for large N

$$M_B = N\left(M + T + \frac{1}{2}V\right) \sim O(N)$$

※1 We can see where the superficial divergence comes from.

If $$M_B = NMf(g, M) = NM(1 + g^2 + \cdots),$$

then

$$e^{-itM_B} = e^{-itNM(1+g^2)}$$

$$= e^{-itN}\left(1 - iMtNg^2 - \frac{1}{2}M^2t^2N^2g^4 + \cdots\right).$$

$\rightarrow$ Each terms will be more and more divergent in N.

Baryons for large N

If quarks are heavy, NR-Schrödinger equation works.

$$H = NM + \sum_i \left(-\frac{\nabla_i^2}{2m}\right) - \frac{g^2}{N}\sum_{i<j}\frac{1}{|x_i - x_j|} \sim O(N)$$

For simplicity, we assume that ...

1) baryons are made from a single flavor (e.g. $\Delta^{++}(uuu)$, $\Omega^-(sss)$)

2) $L=0$ (ground state)

$\Rightarrow$ Since the color is anti-symmetric, a spatial wave fn. is symmetric.

$\rightarrow$ For large N, the mean field approximation will be exact. Many-body wave function can be written as

$$\psi(x_1, \cdots, x_N) = \prod_{i=1}^{N} \phi(x_i)$$

Baryons for large N

【EoM】

Search for a stationary point of $\langle \Psi | (\hat{H} - N\epsilon) | \Psi \rangle$:

$$\langle \Psi | (\hat{H} - N\epsilon) | \Psi \rangle = NM + N \int d^3x \frac{|\nabla \phi|^2}{2M} + \frac{1}{2} N^2 \left(-\frac{g^2}{N}\right) \int d^3x\, d^3y \frac{|\phi(x)|^2 |\phi(y)|^2}{|x-y|}$$
$$+ N\epsilon \int d^3x\, |\phi(x)|^2$$

$$\xrightarrow{0 = \frac{\delta}{\delta \phi} \langle H - N\epsilon \rangle} \quad -\frac{1}{2M} \nabla^2 \phi(x) - g^2 \phi \int d^3y \frac{|\phi(y)|^2}{|x-y|} = \epsilon \phi$$

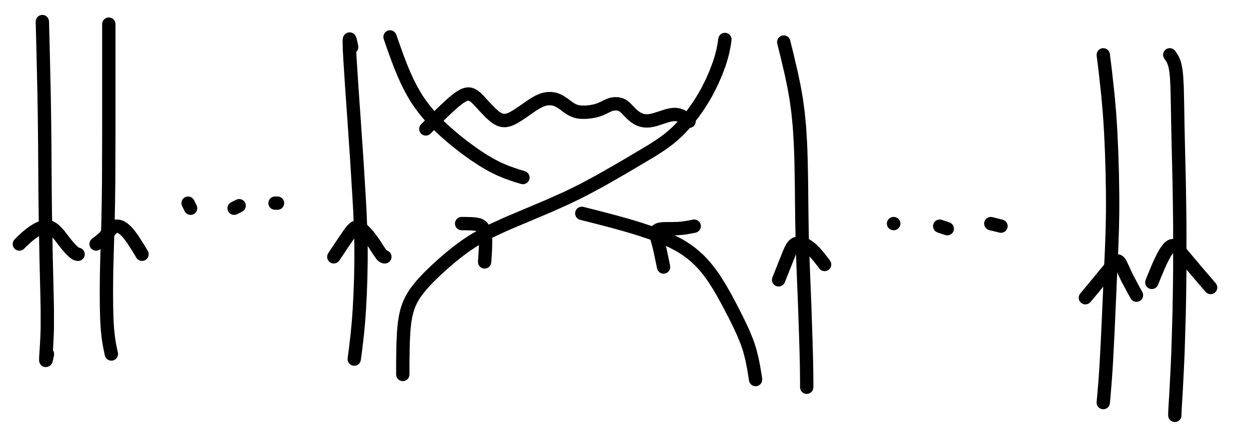
Then, a reduced equation

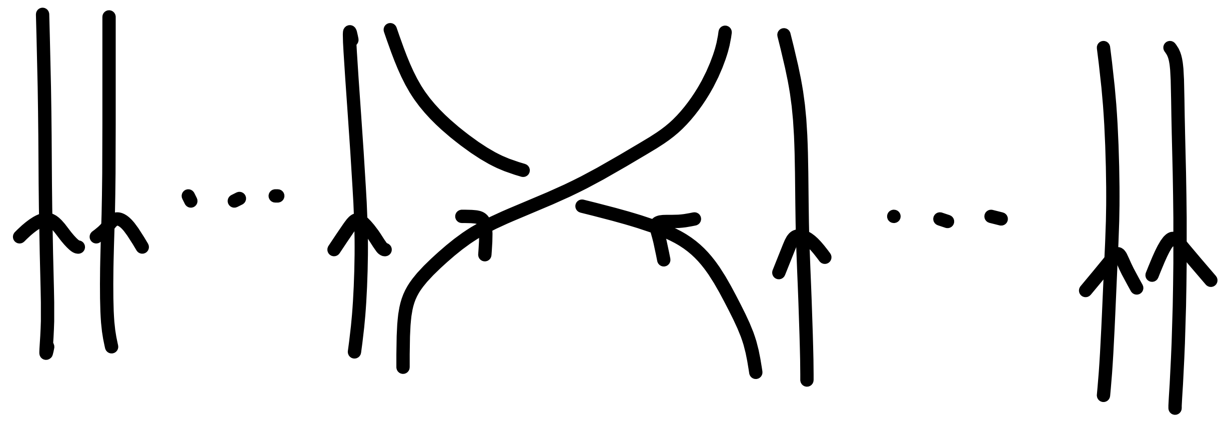
$$-\frac{1}{2M}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{1}{\phi}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\phi\right) + 4\pi g^2 \phi^* \phi = 0$$

can be solved through numerical calculation.

Scattering of Baryons

【 Baryon - Baryon scattering 】

 $\sim \mathcal{O}((1/\sqrt{N})^2 \cdot N^2) = \mathcal{O}(N)$

 $\sim \mathcal{O}(N)$

cf. baryon kinetic energy $T = \frac{1}{2} M_B v^2 \sim \mathcal{O}(N)$

→ This makes a non-trivial large N limit for the scattering cross sections in contrast to mesons.

Scattering of baryons

【 Baryon - Baryon scattering 】

Treat a two-baryon system as a $2N$-quark system.

$$\Psi(x_1, \cdots, x_{2N}; t) = \sum_P (-)^P \prod_{i=1}^{N} \phi_1(x_i, t) \prod_{j=1}^{N} \phi_2(x_j, t)$$

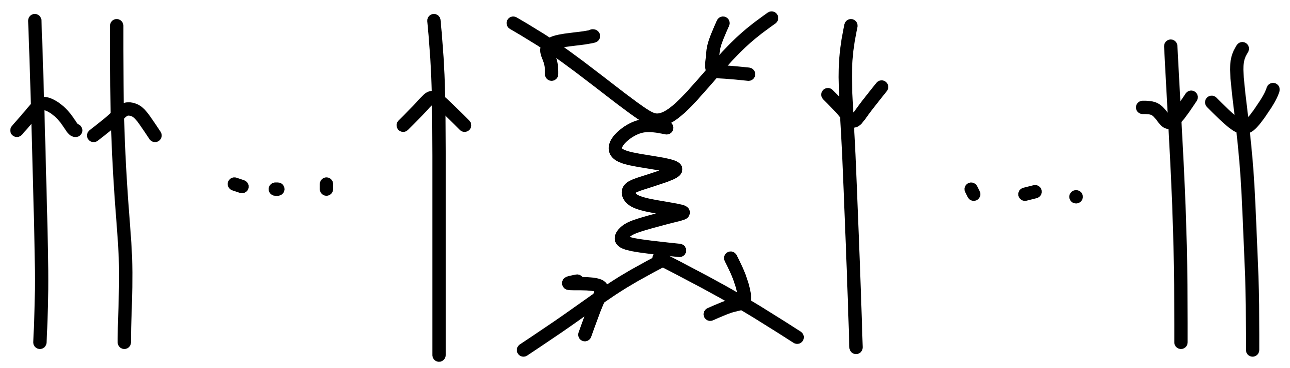
Substitute into $0 = \delta \langle \Psi | (H - i\frac{\partial}{\partial t}) | \Psi \rangle$ to get

$$i \frac{\partial}{\partial t} \phi_1(x,t) = -\frac{\nabla^2}{2M} \phi_1(x,t) - g^2 \phi_1(x,t) \int dy \frac{|\phi_1(y,t)|^2}{|x-y|}$$

$$\underbrace{- g^2 \phi_2(x,t) \int dy \frac{\phi_2^* \phi_1(y,t)}{|x-y|}}_{\hookrightarrow \text{ baryon-baryon interaction}}$$

Scattering of baryons

Baryon - Antibaryon scattering



$$\sim \mathcal{O}\left(\left(1/\sqrt{N}\right)^2 N^2\right) = \mathcal{O}(N)$$

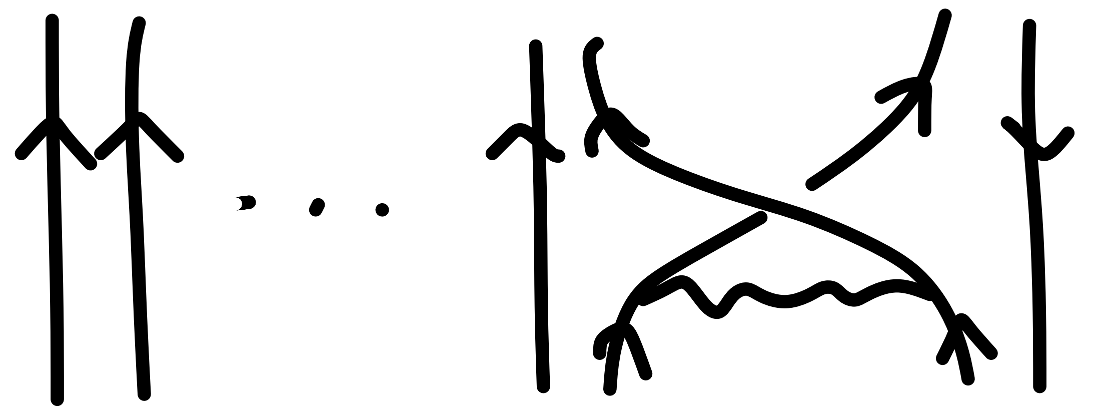
Wave function

$$\Psi(x_1, \cdots, x_N, y_1, \cdots, y_N, t) = \prod_i \underbrace{\phi(x_i, t)}_{\text{baryon}} \prod_j \underbrace{\omega(y_j, t)}_{\text{anti baryon}}$$

→ Can be treated in a same way as a baryon-baryon case.

Scattering of baryons

【 Baryon - Meson scattering 】



$\sim \mathcal{O}\left((1/\sqrt{N})^2 \cdot N \right) = \mathcal{O}(1)$

Comparison with kinetic energy

$\ll T_{\text{baryon}} \sim \mathcal{O}(N)$

$\sim T_{\text{meson}} \sim \mathcal{O}(1)$

Thus,

- baryons → <u>move freely</u> as if there exist no mesons
- mesons → <u>are scattered</u> by baryons .

<u>Wave function</u>

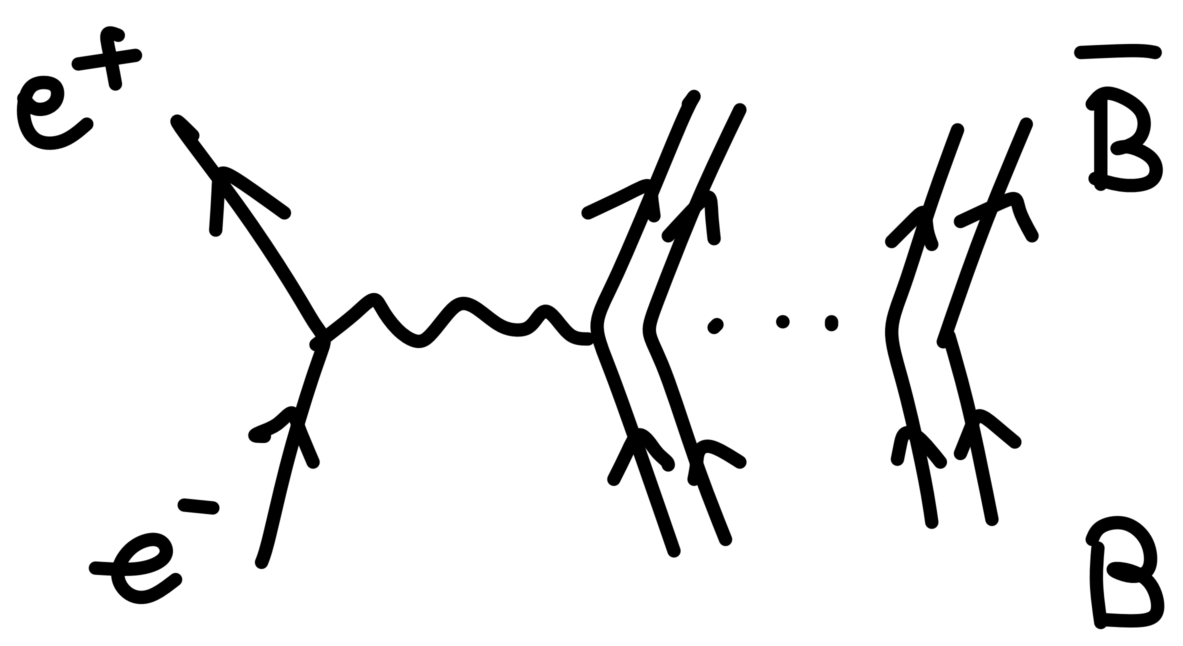
$$\psi(\underbrace{x_1, \cdots, x_{N+1}}_{\text{quarks}}, \underbrace{y}_{\text{anti quarks}}, t) = \sum (-)^P \prod_{i=1}^{N} \phi(x_i, t)\, u(x, y, t)$$

→ <u>Free equations</u> for baryons and <u>equations with baryon background</u> for mesons

"Forbidden processes"

There exist processes that is forbidden in any order of $1/N$.

(i) $e^+e^- \to B\bar{B}$



$e^- e^+ \to \gamma^* \to q\bar{q}$.

One more pair of $q\bar{q}$: probability x

$\Rightarrow$ The probability of creating N pairs of $q\bar{q} \sim x^{N-1}$

Thus, the amplitude of this process $\sim \mathcal{O}(e^{-N})$

$\to$ forbidden

"Forbidden processes"

(ii) $eB \to eB$

If velocity v is fixed, the momentum transfer Q is of order N.

($\because$ $p = M_B v \sim O(N)$)

The matrix element

$$\langle B(v') | J_\mu | B(v) \rangle = \langle q(v') | J_\mu | q(v) \rangle \prod_{i=1}^{N-1} \underbrace{\langle q_i(v') | q_i(v) \rangle}_{\equiv y}$$

$$\sim y^{N-1} = O(e^{-N})$$

(※ One of the quarks is coupled to J_μ, and the rest one not.)

Thus, the amplitude is of order e^{-N}. $\rightarrow$ forbidden

(Crossing symmetry is respected.)

"Forbidden processes"

(iii) $B\bar{B} \to MM$, $MM \to B\bar{B}$ (M: meson)

For the same reason, these two reactions are also suppressed by $O(e^{-N})$.

Short Summary (forbidden processes)

Under the fixed velocities, the reactions such as

$$e^+e^- \to B\bar{B}, \quad eB \to eB$$

$$MM \to B\bar{B}, \quad B\bar{B} \to MM$$

are strongly suppressed by the order $O(e^{-N})$.

Summary and Conclusion

- Mesons for large N
  - Free, stable and non-interacting. No exotics
    $\sim O(1/\sqrt{N})$ $\sim O(1/N)$
  - Mass $\sim O(N^0)$, The number of mesons : infinite
- Baryons for large N
  - Non-trivial interaction with each other $\sim O(N)$
  - Non-interaction with mesons $\sim O(N^0)$
  - The processes involving creating/annihilating of baryons are forbidden.