Notes: Framing anomaly in Chern-Simons theory

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This note covers some basics of the *framing anomaly*, which keeps the pure Chern-Simons theories from being entirely topological at the quantum level. The discussion in this section is based on Witten's papers [1, 2], but we include some additional materials so the line is self-contained [3, 4, 5, 6, 7].

1. The Chern-Simons theory is topological at the classical level, but it is not at the quantum level because it depends on the choices of the two framings: the trivialization of the tangent bundle and the normal bundle of the knots $T\mathcal{M}_3 \oplus NC$ (up to homotopy).

2. The Chern-Simons level shifts by the dual Coxeter number $h^\vee_{\mathfrak{g}} = C_2(\mathfrak{g})/2$, where $C_2(\mathfrak{g})$ is the adjoint Casimir operator, of the gauge group:

$$k \mapsto k + h^\vee_{\mathfrak{g}}. \tag{1}$$

3. The partition function gets the additional contribution from the gravitational η-invariant η_{grav}, which can be made topological by adding the gravitational Chern-Simons term

$$\frac{\dim G}{24}\frac{i}{4\pi}\int_{\mathcal{M}_3} \text{tr}\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega^3\right) \tag{2}$$

as the counter term, where ω is the spin connection on the framed manifold $\mathcal{M}_3$.

4. There is no canonical way to choose the framing (except for the *Atiyah's canonical framing on* S^3). Under the change of the framing $f \mapsto f + s$ (the framing of the tangent bundle $T\mathcal{M}_3$) and $f_a \mapsto f_a + p_a$ (the framing of knot normal bundle $NC = N(\bigcup_a \gamma_a)$), the partition function and the expectation value of the Wilson loops transforms as

$$Z \mapsto \exp\left[2\pi i \frac{c_{\text{WZW}}}{24} s\right] Z \tag{3}$$

$$\langle W_{R_1}(\gamma_1)...W_{R_N}(\gamma_N)\rangle \mapsto \exp\left[2\pi i \sum_{a=1}^{N} h_{R_a} p_a\right] \langle W_{R_1}(\gamma_1)...W_{R_N}(\gamma_N)\rangle\,, \tag{4}$$

where c_{WZW} and h_R is the central charge and the conformal weight of the corresponding primary fields $\Phi_R(z)$ on the boundary Wess-Zumino-Witten CFT.

1 Setup and the intuition on the framing

1.1 Action

We will consider the Chern-Simons theory with the gauge group G defined as

$$S_{\mathrm{CS}}[a] = kI[a] = \frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\left(a\wedge \mathrm{d}a + \frac{2}{3}a^3\right). \tag{5}$$

a is the connection form of the G-bundle that takes its value in $\Omega^1(\mathcal{M}_3, \mathfrak{g})$, the space of one-forms on the three-manifold $\mathcal{M}_3$. To be strict, a is not a globally-defined one-form, but a class of the locally-defined one-forms, each of which is connected via the gauge transformation

$$a \mapsto a' = U(a+d)U^{-1}. \tag{6}$$

Note that the Chern-Simons level k is required to be an integer, considering its gauge invariance. Moreover, it should be an *even* integer on a general manifold, where the second Stiefel-Whitney class $w_2(\mathcal{M}_3)$ does not necessarily vanish, and the spin structure is not necessarily allowed. From now on, we assume that $\mathcal{M}_3$ admits the spin structures (i.e. $w_2(\mathcal{M}_3) = 0$), and that we already specified one spin structure out of them if there are multiple possible choices. In this case, k is allowed to be any integer, and we regard a as a usual connection (not a Spin_C-connection). For simplicity, we assume that the gauge group G is compact and semisimple below.

1.2 Wilson loops and linking numbers

The only observable (gauge-invariant operator) in the theory (5) is the Wilson loops defined on a loop γ as

$$W_R(\gamma) \equiv \mathrm{tr}_R\, \mathcal{P}e^{i\int_\gamma a}, \tag{7}$$

where R labels the integrable highest-weight representation of the Kac-Moody algebra $\widehat{\mathfrak{g}}_k$. This class of representations corresponds to the irreducible representations of $\mathfrak{g}$. For $\mathfrak{g} = su(N)$ for example, the integrable representation of $\widehat{\mathfrak{g}}_k$ is given by the Young tableaux where the number of the boxes in the row is equal to or less than k.

For $G_k = \mathrm{U}(1)_k$, the expectation value of the Wilson loops is given by

$$\langle W_{R_1}(\gamma_1)...W_{R_N}(\gamma_N)\rangle = \exp\left[\frac{2\pi i}{k}\sum_{a,b} n_a n_b \Phi[\gamma_a,\gamma_b]\right], \tag{8}$$

where $\Phi[\gamma_a,\gamma_b]$ is the **linking number**. In a specific local coordinate, it has the explicit representation (the so-called **Polyakov representation**) as

$$\Phi[\gamma_a,\gamma_b] \equiv \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\int_{\gamma_a}\mathrm{d}x_a^\mu \int_{\gamma_b}\mathrm{d}x_b^\nu \frac{(x_a-x_b)^\lambda}{|x_a-x_b|^3}. \tag{9}$$

However, one can immediately notice that $\Phi[\gamma_a,\gamma_b]$ is troublesome if $a=b$, i.e. "self-linking". In this case, since the integral in $\Phi_{\mathrm{self}}[\gamma_a]$ diverges, some regularization prescription is needed. One natural approach is doing "string-splitting" in some sense, just like the point-splitting regularization. In other words, deform a string into a ribbon-like object as in Fig.(1), and treat the

self-linking $\Phi_{\text{self}}[\gamma_a]$ as the linking of the infinitesimally separated two strings $\Phi^f_{\text{self}}[\gamma_a] \equiv \Phi[\gamma_a, \gamma_a^f]$. Given a consistent way to give $\{\gamma_a^f\}$ to each of $\{\gamma_a\}$ (which is referred to as the **framing**), Eq.(8) no longer includes any divergences.

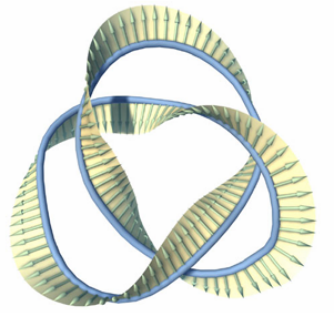

Figure 1: A framed trefoil, specifying one trivialization of the normal bundle NC [7]

However, this choice is not unique, and it is up to us which framing to choose, on which eventually the expectation value of the Wilson loops depends. We can, say, give one more writhe to each loop, which is also another consistent way of the regularization. More formally, we are required to specify a trivialization of the normal bundle on the loops as we will see below.

The punchline is that even though the Chern-Simons theory is topological at the *classial* level, it is not at the *quantum* level. This strange dependence on geometry (more precisely, the metric $g \in \Omega^2(\mathcal{M}_3)$) originates from the functional integral measure: it requires the gauge-fixing condition, which *does* depend on the metric g. This matric dependence corresponds to the fact that the expectation value of Wilson loops $\langle W_{R_1}(\gamma_1)...W_{R_n}(\gamma_n)\rangle$ depends on *how* you embed the loops $C = \{\gamma_1, ..., \gamma_n\}$ in the three-manifold $\mathcal{M}_3$. It requires you to specify the writhes of the loops C (more precisely, which global trivializations of the normal bundle $NC = \coprod_p N_p C$ you choose up to homotopy), which includes the information more than the loops themselves.

2 Semi-classical approximation of Chern-Simons

2.1 Overview

Following Witten's discussion [1, 2] (but presumably in a more modernized way), we consider the semi-classical approach to the Chern-Simons theory given by Eq.(5). Taking the dimensionless parameter k to be the infinity $k \to \infty$ realizes the classical limit because this limit is equivalent to $\hbar \to 0$ for the Boltzmann factor $e^{-\frac{1}{\hbar}S}$ in the functional integral (Or, equivalently, because the quantum group $U_q(\mathfrak{g})$ with $q = e^{2\pi i/k}$ is reduced into the "classical" group G in this limit).

The quantum correction can be captured up to the one-loop order by fluctuating a around the classical configuration A_0 that satisfies the equation of motion

$$F_0 = \mathrm{d}A_0 + A_0 \wedge A_0 = 0. \tag{10}$$

From now on, A_0 is treated as a background gauge field. Substituting $a = A_0 + \tilde{a}$ into Eq.(5), we immediately obtain

$$S_{\text{CS}}[A_0 + \tilde{a}] \simeq kI[A_0] + \frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\,(\tilde{a} \wedge D_{A_0}\tilde{a})\,, \tag{11}$$

where D_{A_0} denotes the covariant derivative in terms of A_0, i.e.

$$D_{A_0}\tilde{a} \equiv \mathrm{d}\tilde{a} + 2A_0 \wedge \tilde{a}. \tag{12}$$

Thus, the total partition function of the problem is reduced into the sum of all the contributions from every single classical configuration as

$$Z \simeq \sum_{A_0} e^{-kI[A_0]} \int \mathcal{D}\tilde{a} \exp\left[-\frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\,(\tilde{a} \wedge D_{A_0}\tilde{a})\right] \tag{13}$$

Accounting for the quantum correction properly (up to at the semiclassical level), we will see the following significant consequences.

2.2 BRST gauge fixing

First, we will fix the gauge following the standard BRST procedure. We introduce the ghost c, the anti-ghost $\bar{c}$, and the Nakanishi-Lautrup field B, all of which take the values in $\Omega^0(\mathcal{M}_3)$ and satisfy under the BRST transformation δ_{BRST}

$$\begin{cases} \delta_{BRST}(\tilde{a}) = D_a c = (\mathrm{d} + a)c \\ \delta_{BRST}(c) = ic^2 \\ \delta_{BRST}(\bar{c}) = iB \\ \delta_{BRST}(B) = 0. \end{cases}$$

Let $F = F(\tilde{a}, c, \bar{c}, B) \in \Omega^3(\mathcal{M}_3; \mathfrak{g})$ be a gauge fixing function taking its value in a $\mathfrak{g}$-valued three-form with the ghost number $N_{\mathrm{FP}} = 0$. The gauge fixing $F = 0$ is done by adding a term

$$\Delta\mathcal{L}_{\mathrm{FP}} = -i\delta_{BRST}\Big(\mathrm{tr}(\bar{c}F)\Big) \tag{14}$$

to the original Lagrangian. In the following, we will take the Lorentz gauge

$$F \equiv D_{A_0}(\star\tilde{a}) = g^{\mu\nu}(D_{A_0})_\mu \tilde{a}_\nu \ \mathrm{vol}(g) = 0. \tag{15}$$

This gauge fixing condition involves the Hogde star operator $\star : \Omega^1(\mathcal{M}_3) \to \Omega^2(\mathcal{M}_3)$ and the volume three form $\mathrm{vol}(g) = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$ for some fixed local coordinate. Then, paying attention to the fact that δ_{BRST} itself is Grassmanian-odd as well, we get the gauge fixing term

$$\begin{aligned} \Delta\mathcal{L}_{\mathrm{FP}} &= -i\delta_{BRST}\Big(\mathrm{tr}(\bar{c}D_{A_0}(\star\tilde{a}))\Big) \\ &= -i\,\mathrm{tr}\left(\delta_{BRST}(\bar{c}\mathrm{d}\star\tilde{a} + 2\bar{c}A_0 \wedge \star\tilde{a})\right) \\ &= \mathrm{tr}\left(B(D_{A_0}(\star\tilde{a})) + i\bar{c}D_{A_0}(\star D_a c)\right). \end{aligned} \tag{16}$$

Therefore, the total action reads

$$kI[A_0] + \frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\,(\tilde{a} \wedge D_{A_0}\tilde{a}) + \int_{\mathcal{M}_3} \mathrm{tr}(BD_{A_0} \star \tilde{a} + i\bar{c}D_{A_0} \star D_{A_0}c), \tag{17}$$

where we use the fact that A_0 is a classical configuration and $\tilde{a}$-linear term should be dropped to replace D_a with D_{A_0} in the last term of $\Delta\mathcal{L}$.

2.3 Functional integral and its regularization

The functional integral of the ghosts $\mathcal{D}c\mathcal{D}\bar{c}$ is easy to do and we get the functional determinant

$$\mathrm{Det}\Delta_0 = \mathrm{Det}(\star D_{A_0} \star D_{A_0}). \tag{18}$$

We denote $\star D_{A_0} \star D_{A_0}$ as Δ_0 because it is the standard Laplacian operating on forms. Note that the eigenvalues of Δ_0 is real and positive because Δ_0 is positive and self-adjoint.

More complicated is the functional integral for the terms left. First, for later convenience, we introduce a three-form field $\phi \equiv (2\pi/k) \star B \in \Omega^3(\mathcal{M}_3)$. Then, the terms left to be integrated are

$$\begin{aligned}
\frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\left(\tilde{a}\wedge D_{A_0}\tilde{a} + 2(\star\phi)D_{A_0}\star\tilde{a}\right) &= \frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\left(\tilde{a}\wedge D_{A_0}\tilde{a} + 2\phi\star D_{A_0}\star\tilde{a}\right) \\
&= \frac{ik}{4\pi}\int_{\mathcal{M}_3} \mathrm{tr}\left((\tilde{a}+\phi)\wedge\star(\star D_{A_0} + D_{A_0}\star)(\tilde{a}+\phi)\right) \\
&\equiv \frac{i}{2}\left\langle\Phi, L\Phi\right\rangle, \qquad (19)
\end{aligned}$$

where $\Phi \equiv \sqrt{k/2\pi}(\tilde{a}+\phi) \in \Omega^{\mathrm{odd}}(\mathcal{M}_3) = \Omega^1(\mathcal{M}_3)\oplus\Omega^3(\mathcal{M}_3)$ and $L \equiv \star D_{A_0} + D_{A_0}\star$. From now on, we use the inner product notation $\langle\omega,\eta\rangle \equiv \int_{\mathcal{M}_3}\mathrm{tr}(\omega\wedge\star\eta)$ here, which is defined for $\omega,\eta\in\Omega^k(\mathcal{M}_3;\mathfrak{g})$.

L is an operator of the kind called the **elliptic operator**, the structure of whose eigenvalues can be captured within the framework of the **Atiyah-Patodi-Singer (APS) index theorem** and the η-invariant. Now, L is a self-adjoint operator, which is a map either from $\Omega^{\mathrm{odd}}(\mathcal{M}_3)$ to $\Omega^{\mathrm{odd}}(\mathcal{M}_3)$ or from $\Omega^{\mathrm{even}}(\mathcal{M}_3)$ to $\Omega^{\mathrm{even}}(\mathcal{M}_3)$. Let us denote the restriction of L on $\Omega^{\mathrm{even}}(\mathcal{M}_3)$ and $\Omega^{\mathrm{odd}}(\mathcal{M}_3)$ as L_+ and L_-, respectively. We are now interested in

$$L_- : \Omega^{\mathrm{odd}}(\mathcal{M}_3) \to \Omega^{\mathrm{odd}}(\mathcal{M}_3). \tag{20}$$

instead of L_+. Using the diagonalized basis of L_-, the functional integral will be reduced into

$$\int_{\Omega^{\mathrm{odd}}(\mathcal{M}_3)}\mathcal{D}\Phi\exp\left[-\frac{i}{2}\left\langle\Phi, L_-\Phi\right\rangle\right] = \prod_i\int_{-\infty}^{\infty}\frac{\mathrm{d}h_i}{\sqrt{2\pi}}\exp\left[-\frac{i}{2}\lambda_i h_i^2\right] \tag{21}$$

where $\{\lambda_i\}$ is the spectrum of the operator L_-:

$$\{\lambda_i\} = \mathrm{Spec}(L_-;\Omega^{\mathrm{odd}}(\mathcal{M}_3)). \tag{22}$$

The problem here is that the integration oscillates and does not converge.

To tame this divergence, we introduce the Yang-Mills term

$$\begin{aligned}
S_{\mathrm{YM}} &= \frac{1}{2g^2}\int_{\mathcal{M}_3} f\wedge\star f \\
&\simeq \frac{1}{2g^2}\int_{\mathcal{M}_3} D_{A_0}\tilde{a}\wedge\star D_{A_0}\tilde{a} \\
&= \frac{1}{2g^2}\left\langle\Phi, D_{A_0}^\dagger D_{A_0}\Phi\right\rangle \qquad (23)
\end{aligned}$$

as a regularization factor. Here, we drop the higher power of $\tilde{a}$ in $f = D_a a \simeq D_{A_0}A_0 + D_{A_0}\tilde{a} = D_{A_0}\tilde{a}$ and the linear terms, and use the $D_{A_0}\Phi = D_{A_0}(\tilde{a}+\phi) = D_{A_0}\tilde{a}$ in the last line. Even though the

Yang-Mills term is irrelevant in the three-dimension in the IR limit $g^2 \to \infty$, Eq.(23) regulates the divergence in Eq.(21). We denote the spectrum of the operator $D^\dagger_{A_0} D_{A_0}$ on $\Omega^{\rm odd}(\mathcal{M}_3)$ as

$$\{\mu_i^2\} = {\rm Spec}(D^\dagger_{A_0} D_{A_0}; \Omega^{\rm odd}(\mathcal{M}_3)), \tag{24}$$

and then Eq.(21) now converges properly, and it is modified to

$$\begin{aligned}
\prod_j \int_\infty^\infty \frac{{\rm d}h_j}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{1}{g^2}\mu_j^2 + i\lambda_j\right) h_j^2\right]\Bigg|_{g^2\to\infty} &= \prod_j \frac{1}{\sqrt{\mu_j^2/g^2 + i\lambda_j^{(1)}}}\Bigg|_{g^2\to\infty} \\
&= \prod_j \frac{1}{\sqrt{|\lambda_j|}} e^{\frac{i\pi}{4}{\rm sgn}(\lambda_j)} \\
&= \left(\prod_j \frac{1}{\sqrt{|\lambda_j|}}\right) e^{\frac{i\pi}{4}\sum_j {\rm sgn}(\lambda_j)}.
\end{aligned} \tag{25}$$

2.4 Spectrum ζ-function regularization and η-invariant

The summation $\sum_j {\rm sgn}(\lambda_j)$ in the last line needs to be regularized because it is, just naively, (# positive eigenvalues) $-$ (# negative eigenvalues) $= \infty - \infty$. The standard method to do this is using the analytic continuation of the spectrum ζ-function defined as

$$\begin{aligned}
\zeta(L_-; s) &\equiv \sum_j \left(\frac{\mu_j^2}{g^2} + i\lambda_j\right)^{-s}\Bigg|_{g^2\to\infty} \\
&= \sum_j \exp\left[s\left(-\ln|\lambda_j| + \frac{i\pi}{2}{\rm sgn}(\lambda_j)\right)\right].
\end{aligned} \tag{26}$$

Note that the imaginary part of $\lim_{s\to 0}\zeta'(L_-;s)$ is formally equal to $(\pi/2)\sum_j {\rm sgn}(\lambda_j)$. Since the ζ-function can be analytically continued to the whole complex plane except for the finite points, $\lim_{s\to 0}\zeta'(L_-;s)$ can be defined properly although $\sum_j {\rm sgn}(\lambda_j)$ makes no sense. As such, we define **η-invariant** of the operator $\star D_{A_0}$ as

$$\eta[A_0, g] \equiv \frac{1}{\pi}\lim_{s\to 0}\zeta'(L_-;s) \overset{\text{formally}}{=} \frac{1}{2}\sum_j {\rm sgn}(\lambda_j), \tag{27}$$

showing the dependence on A_0 and the metric g (via the Hodge star operator) explicitly. Therefore, Eq.(25) is

$$\frac{1}{\sqrt{|{\rm Det}(L_-)|}} e^{\frac{i\pi}{2}\eta[A_0,g]}, \tag{28}$$

and the total partition function (13) reads

$$Z \simeq \sum_{A_0} T[A_0] e^{\frac{i\pi}{2}\eta[0,g] - kI[A_0]}, \tag{29}$$

where the real-valued quantity

$$T[A_0] \equiv \frac{{\rm Det}\Delta_0}{\sqrt{|{\rm Det}(L_-)|}} \tag{30}$$

is called the **Ray-Singer torsion** of a given flat connection A_0 (it is, importantly, a topological invariant).

2.5 Gravitational Chern-Simons as a counter term

The APS index theorem provides an additional relation between the η-invariant and the Chern-Simons term as

$$\frac{\pi}{2}\Big(\eta[A_0,g]-\eta[0,g]\Big)=h^\vee_{\mathfrak{g}}I[A_0]. \tag{31}$$

Here $h^\vee_{\mathfrak{g}}$ is the **dual Coxeter number** of the gauge group, which is related to the adjoint Casimir operator $C_2(\mathfrak{g})$ as $h^\vee_{\mathfrak{g}}=C_2(\mathfrak{g})/2$. Using this relation, the total partition function reads

$$Z\simeq e^{\frac{i\pi}{2}\eta[0,g]}\sum_{A_0}T[A_0]\exp\Big[-(k+h^\vee_{\mathfrak{g}})I[A_0]\Big]. \tag{32}$$

The only ingredient of Eq.(32) that is *not* topologically invariant is $\eta[0,g]$. Because L_- is, if $A_0=0$, reduced into the direct sum of dim$G$ copies of $\mathfrak{D}_-\equiv(\mathrm{d}\star+\star\,\mathrm{d})_{\mathrm{odd}}$, $\eta[0,g]$ is equivalent to dim$G$ times as large as η_{grav}, the η-invariant of $\mathfrak{D}_-$, i.e. $\eta[0,g]=(\mathrm{dim}G)\eta_{\mathrm{grav}}$. Therefore, the overall phase factor in Eq.(32) is

$$\exp\left[\frac{i\pi}{2}\eta[0,g]\right]=\exp\left[\frac{i\pi(\mathrm{dim}G)}{2}\eta_{\mathrm{grav}}\right]. \tag{33}$$

This metric dependence, unfortunately but as expected, cannot be eliminated. We must inevitably encounter this breaking of the topologicalness if we attempt to make the original classical theory quantum. This breaking due to the quantization is often referred to as the **framing anomaly** of the Chern-Simons theory.

Eq.(33) depends only on the geometry, and its dependence can be canceled by adding the *gravitational Chern-Simons theory*

$$I[g]\equiv\frac{i}{4\pi}\int_{\mathcal{M}_3}\mathrm{tr}\left(\omega\mathrm{d}\omega+\frac{2}{3}\omega^3\right) \tag{34}$$

to the partition function just like counter terms in the renormalization procedure, where ω is the Levi-Civita connection on the spinor bundle

$$\mathrm{Spin}(3)\simeq\mathrm{SU}(2)\hookrightarrow P\to\mathcal{M}_3. \tag{35}$$

This is because the Atiyah-Singer index theorem argues that the sum

$$\frac{i}{2}\eta_{\mathrm{grav}}+\frac{1}{12}\frac{1}{2\pi}I[g] \tag{36}$$

is a topological invariant in total, while it is not in each term. Therefore, adding the term

$$\exp\left[\frac{(\mathrm{dim}G)}{2\cdot12}I[g]\right]=\exp\left[\frac{i(\mathrm{dim}G)}{96\pi}\int_{\mathcal{M}_3}\mathrm{tr}\left(\omega\mathrm{d}\omega+\frac{2}{3}\omega^3\right)\right] \tag{37}$$

to the partition function (32) completely cancels g dependence in Eq.(33), and we can obtain a fully topological-invariant representation of this quantum theory. Finally, we reach a topological invariant one-loop partition function

$$Z=\exp\left[i\pi(\mathrm{dim}G)\left(\frac{1}{2}\eta_{\mathrm{grav}}-\frac{i}{24\pi}I[g]\right)\right]\sum_{A_0}T[A_0]\exp\Big[-(k+h^\vee_{\mathfrak{g}})I[A_0]\Big]. \tag{38}$$

Note that the partition function is well-defined on a **framed manifold**, i.e. a manifold with one specific choice of the framing. Here, a framing means a trivialization of the tangent bundle $\mathrm{T}\mathcal{M}_3 = \coprod_p \mathrm{T}_p\mathcal{M}_3$ on $\mathcal{M}_3$, and it is known that such trivialization always exists for an arbitrary three-manifold. Once one specifies a trivialization of $\mathrm{T}\mathcal{M}_3$, it automatically and uniquely determines a spin structure of $\mathcal{M}_3$. This is the reason why the gravitational Chern-Simons term makes sense if and only if a trivialization of $\mathrm{T}\mathcal{M}_3$ is specified.

3 Choices of framing

The problem is, while Eq.(38) depends on the framing, we have no canonical choice of the framing.

In other words, any Levi-Civita connections seem to have the torsion from the other specific spin structure and vice versa. Everything is relative here!

The best way to see how the theory depends on the framing is to see how it transforms under the change of the framings. Analogous to the Chern-Simons case, the framings form a $\mathbb{Z}$-torsor ($\mathbb{Z}$-principle homogeneous space). In other words, the difference between two different framings is labeled by an integer $s \in \mathbb{Z}$, while the framings themselves have no "identity". As such, under the change of framings $f \mapsto f + s$ ($s \in \mathbb{Z}$), the transformation law reads $I[g] \mapsto I[g] + 2\pi i s$. This resembles the Chern-Simons transformation law $I[a] \mapsto I[a] + 2\pi i n w$ ($n \in \mathbb{Z}$ and $w \in \pi_3(G) \simeq \mathbb{Z}$). Under the framing change, Eq.(38) transforms as

$$Z \mapsto Z' = \exp\left[2\pi i \frac{\mathrm{dim}G}{24} s\right] Z. \tag{39}$$

From the CS/WZW-correspondence perspective, it is expected to be [1]

$$Z \mapsto Z' = \exp\left[2\pi i \frac{c_{\mathrm{WZW}}}{24} s\right] Z, \tag{40}$$

where c_{WZW} is the central charge of the boundary CFT given by

$$c_{\mathrm{WZW}} = \frac{k(\mathrm{dim}G)}{k + h^\vee_{\mathfrak{g}}} \overset{k\to\infty}{\longrightarrow} \mathrm{dim}G. \tag{41}$$

Moreover, the framings of the knots should also be mentioned. Just like that of the spinor bundle, the change of the framing f_a of the loop γ_a is labeled by integers as $\{f_a\} \mapsto \{f_a + p_a\}$ ($p_a \in \mathbb{Z}$). In this case, the expectation value of Wilson loops transforms as

$$\begin{aligned}
\mathrm{U}(1)_k \text{ case: } \langle W_{n_1}(\gamma_1)...W_{n_N}(\gamma_N)\rangle &\mapsto \exp\left[\frac{i\pi}{k}\sum_{a=1}^{N} n_a^2 p_a\right] \langle W_{n_1}(\gamma_1)...W_{n_N}(\gamma_N)\rangle \\
G \text{ case: } \langle W_{R_1}(\gamma_1)...W_{R_N}(\gamma_N)\rangle &\mapsto \exp\left[2\pi i \sum_{a=1}^{N} h_{R_a} p_a\right] \langle W_{R_1}(\gamma_1)...W_{R_N}(\gamma_N)\rangle\,,
\end{aligned} \tag{42}$$

where h_R is the conformal weight of a WZW operator corresponding to the Wilson loop $W_R(\gamma)$ defined as

$$h_R = \frac{C_R(\mathfrak{g})}{2(k + h^\vee_{\mathfrak{g}})}. \tag{43}$$

[1] Why? -Yuto

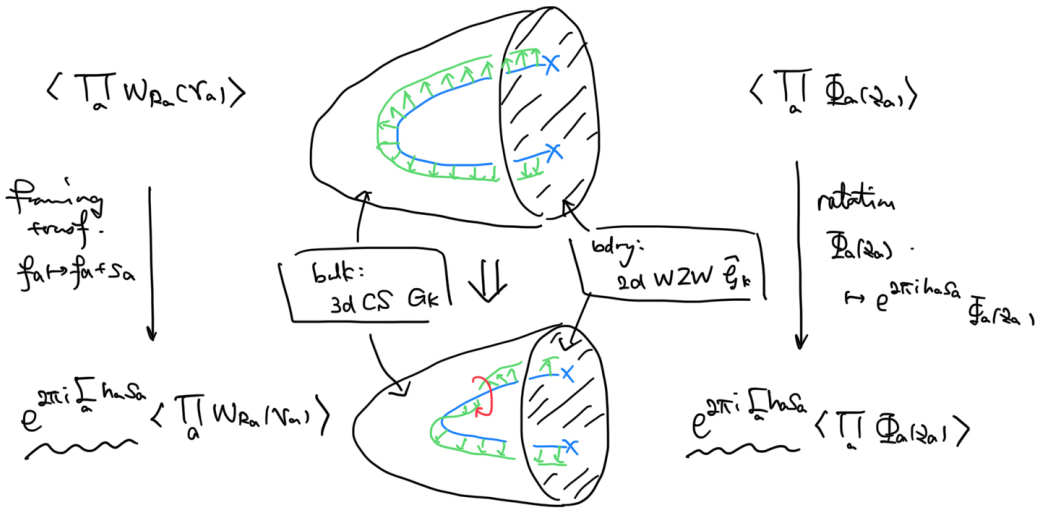

Figure 2: CS/WZW-correspondence from the perspective of twist.. Giving one more writhe to a Wilson loop corresponds to rotating a primary field in the boundary CFT, both of which yield the same phase.

This surprising relationship is natural if we think this way shown in Fig.(2): Giving a twist to a Wilson loop $W_R(\gamma)$, which ends at a point on the boundary, is rotating a primary field $\Phi_a(z_a)$ living in the boundary by 2π, and these operations yield the identical phase $e^{2\pi i h_a}$.

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