Dec 11th, 2024

FRADKIN-SHENKER CONTINUITY IN GAUGE THEORIES

Stat-Mech Final presentation

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E. Fradkin and S. H. Shenker. *Phase diagrams of lattice gauge theories with Higgs fields.* Phys. Rev. D **19,** 3682 (1979)

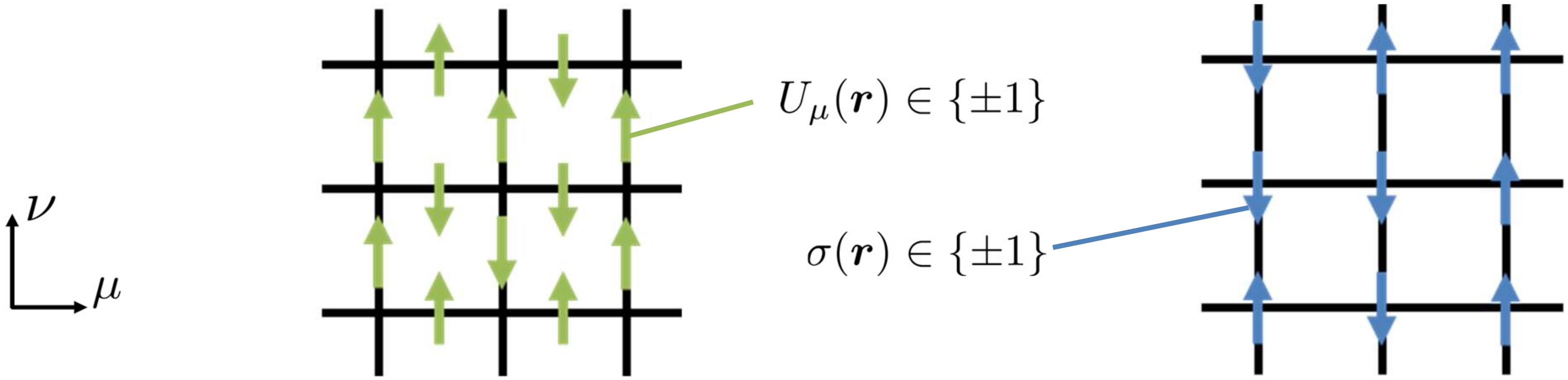
[REVIEW] $\mathbb{Z}_2$ gauge theory with matters

In *problem set 4*, we played with a matter-coupled $\mathbb{Z}_2$ gauge theory:

$$S_{\mathbb{Z}_2}[U_\mu(\boldsymbol{r}), \sigma(\boldsymbol{r})] = K \sum_{(\boldsymbol{r},\mu,\nu)} \square_{\mu\nu}(\boldsymbol{r}) + \beta \sum_{(\boldsymbol{r},\mu)} \sigma(\boldsymbol{r}) U_\mu(\boldsymbol{r}) \sigma(\boldsymbol{r}+\boldsymbol{e}_\mu)$$

Spin variable: $U_\mu(\boldsymbol{r}), \sigma(\boldsymbol{r}) \in \{\pm 1\}$

Plaquette: $\square_{\mu\nu}(\boldsymbol{r}) \equiv U_\mu(\boldsymbol{r}) U_\nu(\boldsymbol{r}+\boldsymbol{e}_\mu) U_\mu(\boldsymbol{r}+\boldsymbol{e}_\nu) U_\nu(\boldsymbol{r})$



[REVIEW] $\mathbb{Z}_2$ gauge theory with matters

$$S_{\mathbb{Z}_2}[U_\mu(\boldsymbol{r}), \sigma(\boldsymbol{r})] = K \sum_{(\boldsymbol{r},\mu,\nu)} \Box_{\mu\nu}(\boldsymbol{r}) + \beta \sum_{(\boldsymbol{r},\mu)} \sigma(\boldsymbol{r}) U_\mu(\boldsymbol{r}) \sigma(\boldsymbol{r}+\boldsymbol{e}_\mu)$$

Two simple cases:

(a) $K = \infty$ ⇨ $S_{\mathrm{Ising}} = \beta \sum_{(\boldsymbol{r},\mu)} \sigma(\boldsymbol{r})\sigma(\boldsymbol{r}+\boldsymbol{e}_\mu)$ since $^\forall \Box_{\mu\nu}(\boldsymbol{r}) = 1$

has an **ordered-disordered** phase transition

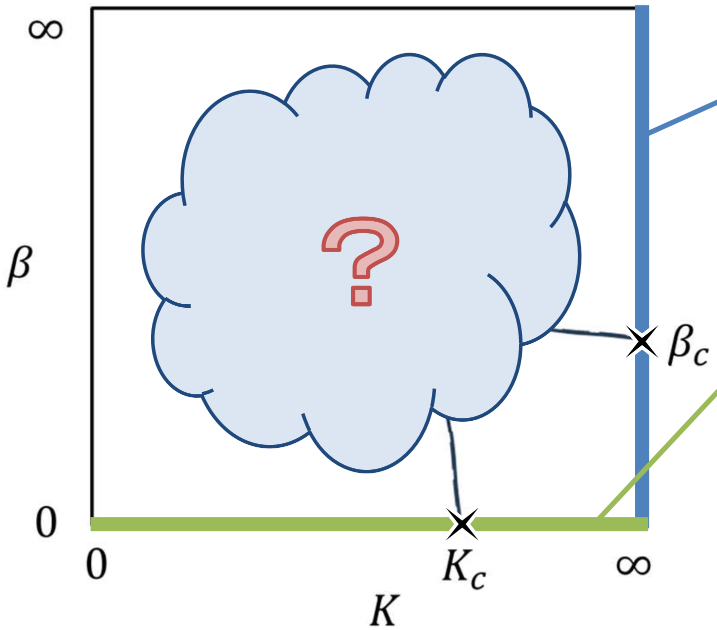
(b) $\beta = 0$ ⇨ $S_{\mathrm{pure}} = K \sum_{(\boldsymbol{r},\mu,\nu)} \Box_{\mu\nu}(\boldsymbol{r})$

has a **confinement-deconfinement** phase transition

[REVIEW] $\mathbb{Z}_2$ gauge theory with matters

$$S_{\mathbb{Z}_2}[U_\mu(\boldsymbol{r}),\sigma(\boldsymbol{r})] = K \sum_{(\boldsymbol{r},\mu,\nu)} \Box_{\mu\nu}(\boldsymbol{r}) + \beta \sum_{(\boldsymbol{r},\mu)} \sigma(\boldsymbol{r}) U_\mu(\boldsymbol{r}) \sigma(\boldsymbol{r}+\boldsymbol{e}_\mu)$$

The conjectured phase diagram:



Ising model

$$S_{\text{Ising}} = \beta \sum_{(\boldsymbol{r},\mu)} \sigma(\boldsymbol{r})\sigma(\boldsymbol{r}+\boldsymbol{e}_\mu)$$

Pure gauge theory

$$S_{\text{pure}} = K \sum_{(\boldsymbol{r},\mu,\nu)} \Box_{\mu\nu}(\boldsymbol{r})$$

Q: What do other areas look like?

Today's goal & Summary

Main Question:

How many phases does the **gauge-Higgs model** have in general?

$K = \infty$ → Ising model

$\beta = 0$ → Pure gauge theory

Main Results:

# of phases is determined by **SSB pattern** (i.e. irrep. of Higgs)

1. Fundamental Higgs (SSB: $G \to 1$)

   ⇨ **Two phases** (Coulomb, Higgs-confinement)

2. Higher irrep. Higgs (SSB: $G \to H(\subset G)$)

   ⇨ **Three phases** (Coulomb, Higgs, Confinement)

Lattice-formulation of gauge theory

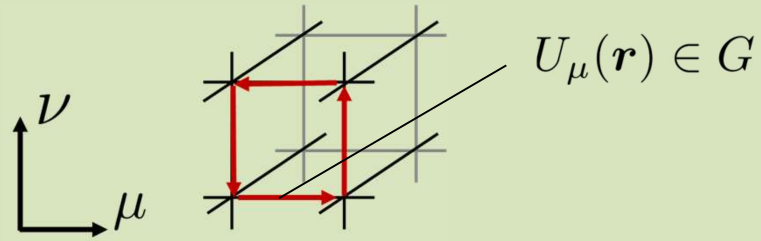
The general form of a G-gauge theory coupled with Higgs

$$S[U_\mu(\boldsymbol{r}),\phi(\boldsymbol{r})] = \frac{K}{2}\sum_{(\boldsymbol{r},\mu,\nu)} \mathrm{Tr}[\Box_{\mu\nu}(\boldsymbol{r}) + \mathrm{h.c.}] + \frac{\beta}{2}\sum_{(\boldsymbol{r},\mu)} [\phi(\boldsymbol{r})D(U_\mu(\boldsymbol{r}))\phi^\dagger(\boldsymbol{r}+\boldsymbol{e}_\mu) + \mathrm{c.c.}]$$

Gauge Sector

$\Box_{\mu\nu}(\boldsymbol{r})$ is a **plaquette** defined as:

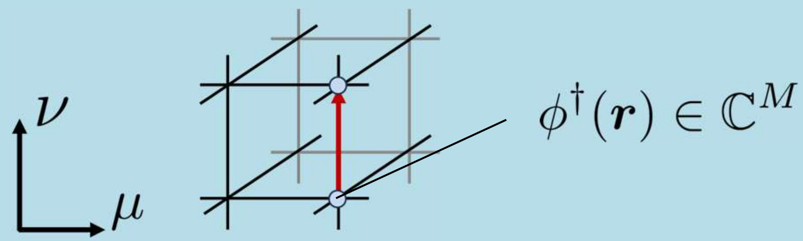
$$\Box_{\mu\nu}(\boldsymbol{r}) \equiv U_\mu(\boldsymbol{r})U_\nu(\boldsymbol{r}+\boldsymbol{e}_\mu)U_\mu^\dagger(\boldsymbol{r}+\boldsymbol{e}_\nu)U_\nu^\dagger(\boldsymbol{r})$$



Higgs Sector

Higgs obeys M-dim irrep. D:

$$D : G \to \mathrm{Aut}(\mathbb{C}^M)$$



Gauge redundancy

This model has a G -gauge redundancy with $V(\boldsymbol{r}) \in G$:

$$U_\mu(\boldsymbol{r}) \mapsto V(\boldsymbol{r})U_\mu(\boldsymbol{r})V^{-1}(\boldsymbol{r}+\boldsymbol{e}_\mu), \qquad \phi^\dagger(\boldsymbol{r}) \mapsto D(V(\boldsymbol{r}))\phi^\dagger(\boldsymbol{r})$$

e.g.) $G = \mathrm{U}(1)$ with a charge $q \in \mathrm{Hom}(\mathrm{U}(1), \mathrm{U}(1)) \simeq \mathbb{Z}$

Dynamical variables:

$$U_\mu(\boldsymbol{r}) = e^{iA_\mu(\boldsymbol{r})}, \quad \phi(\boldsymbol{r}) = e^{i\theta(\boldsymbol{r})}, \quad D(U_\mu(\boldsymbol{r})) = e^{iqA_\mu(\boldsymbol{r})}$$

Gauge transformation $V(\boldsymbol{r}) = e^{-i\alpha(\boldsymbol{r})}$:

$$A_\mu(\boldsymbol{r}) \mapsto A_\mu(\boldsymbol{r}) + \partial_\mu\alpha, \qquad \theta(\boldsymbol{r}) \mapsto \theta(\boldsymbol{r}) + q\alpha(\boldsymbol{r})$$

The two extreme cases (1)

(a) $K=\infty$: reduced to a **Higgs-Heisenberg model**

$$\frac{K}{2}\sum_{(\boldsymbol{r},\mu,\nu)}\mathrm{Tr}[\Box_{\mu\nu}(\boldsymbol{r})+\mathrm{h.c.}] \xrightarrow{K=\infty}$$

Constraints:

$$^{\forall}\Box_{\mu\nu}(\boldsymbol{r})=1_G$$

$$\therefore {}^{\forall}\, U_\mu(\boldsymbol{r})=1_G$$

(assume a manifold with trivial topology)

The action is reduced to

$$\boxed{S_{K=\infty}[U_\mu(\boldsymbol{r}),\phi(\boldsymbol{r})]=\frac{\beta}{2}\sum_{(\boldsymbol{r},\mu)}[\phi(\boldsymbol{r})\phi^\dagger(\boldsymbol{r})+\mathrm{c.c.}]}$$

For a sufficiently large dimension, a transition exists at $\beta=\beta_c$.

The two extreme cases (1)

(a) $K = \infty$: reduced to a **Higgs-Heisenberg model**

$\beta > \beta_c$: Ordered phase (massless correlation)

$$\left\langle \phi(0)\phi^\dagger(\boldsymbol{r}) \right\rangle \sim \langle \phi \rangle^2$$ (G : discrete)

$$\left\langle \phi(0)\phi^\dagger(\boldsymbol{r}) \right\rangle \sim \langle \phi \rangle^2 \text{const.} \times \exp\left[\frac{\text{const.}}{|\boldsymbol{r}|^{d-2}}\right]$$ (G : continuous)

Contribution from **spin wave**

$\beta < \beta_c$: Disordered phase (massive correlation)

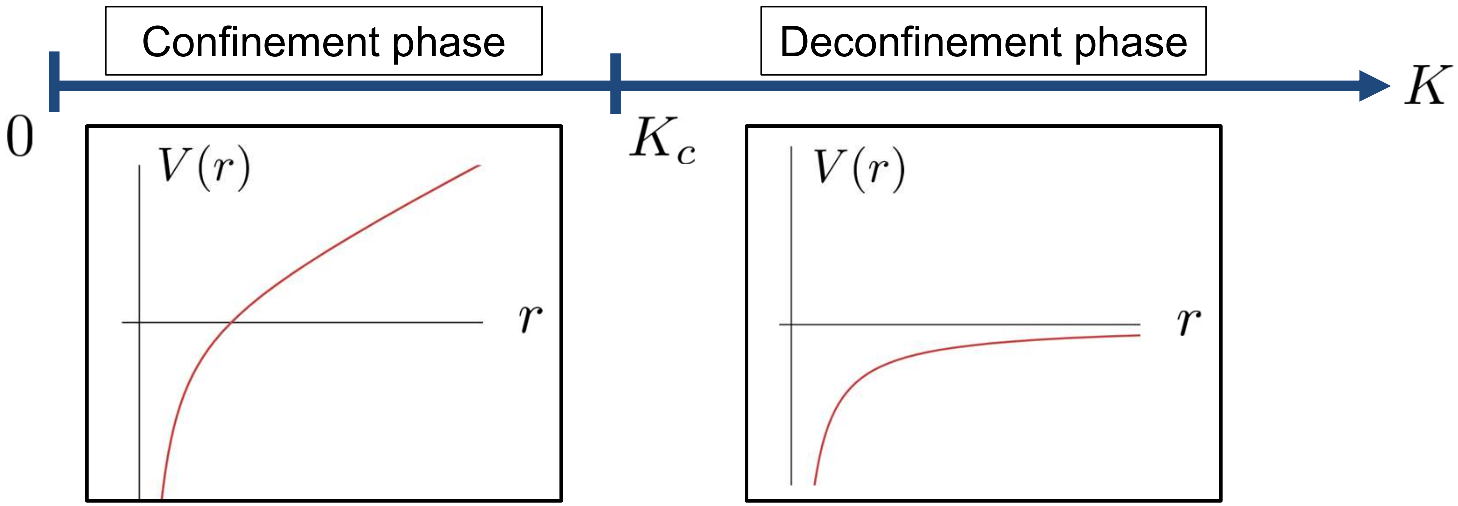
$$\left\langle \phi(0)\phi^\dagger(\boldsymbol{r}) \right\rangle \sim \text{const.} \times e^{-\frac{|\boldsymbol{r}|}{\xi}}$$

The two extreme cases (2)

(b) $\beta = 0$: reduced to a **pure gauge theory**

$$\boxed{S_{\beta=0}[U_\mu(\boldsymbol{r}), \phi(\boldsymbol{r})] = \frac{K}{2} \sum_{(\boldsymbol{r},\mu,\nu)} \mathrm{Tr}[\Box_{\mu\nu}(\boldsymbol{r}) + \mathrm{h.c.}]}$$

For a sufficiently large dimension, a transition exists at $K = K_c$.

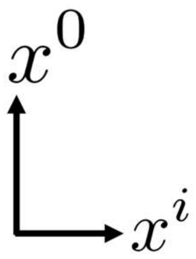


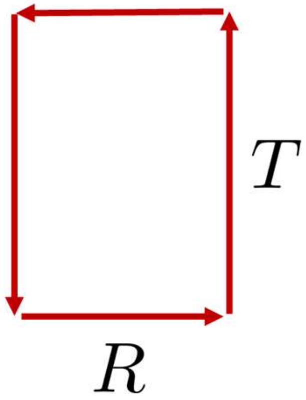
The two extreme cases (2)

(b) $\beta = 0$: reduced to a **pure gauge theory**

Fundamental Wilson loop is an order parameter of confinement:

$$\boxed{W_\Gamma = \left\langle \mathrm{Tr}_{(\mathrm{Fund})} \left[\prod_{(\boldsymbol{r},\mu)\in\Gamma} U_\mu(\boldsymbol{r}) \right] \right\rangle \sim e^{-V(R)T}}$$





$K < K_c$: Confinement phase

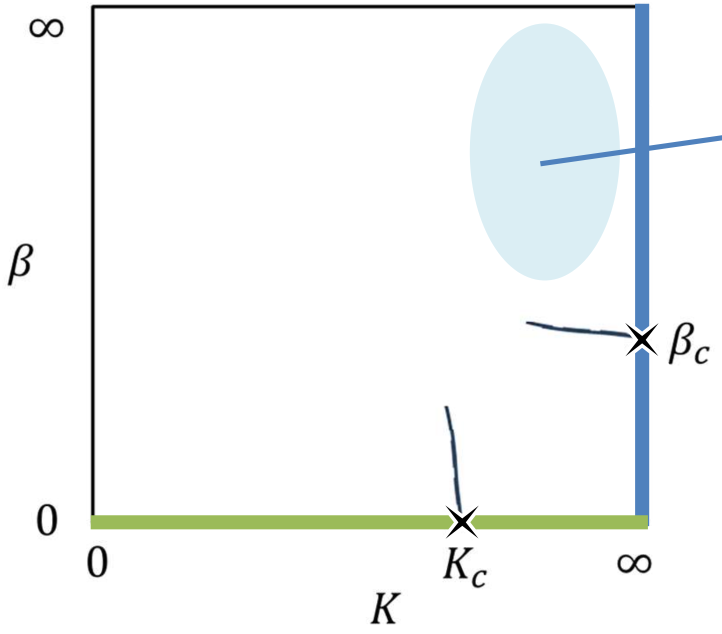
$$W_\Gamma \sim e^{-\mathrm{Area}(\Gamma)} \quad \text{(Area law)}$$

$K > K_c$: Deconfinement phase

$$W_\Gamma \sim e^{-\mathrm{Perimeter}(\Gamma)} \quad \text{(Perimeter law)}$$

Other regions (Higgs phase)

The conjectured phase diagram:



Higgs phase

1. non-trivial VEV: $\langle \sigma(\boldsymbol{r}) \rangle \neq 0$
2. Yukawa-like decaying 2-point function

$$\left\langle \sigma(0) \prod_{(\boldsymbol{r},\mu)\in\Gamma} U_\mu(\boldsymbol{r})\sigma(\boldsymbol{r}) \right\rangle \sim e^{-2|\boldsymbol{r}|\exp[-8K-2\beta]} \equiv e^{-m|\boldsymbol{r}|}$$

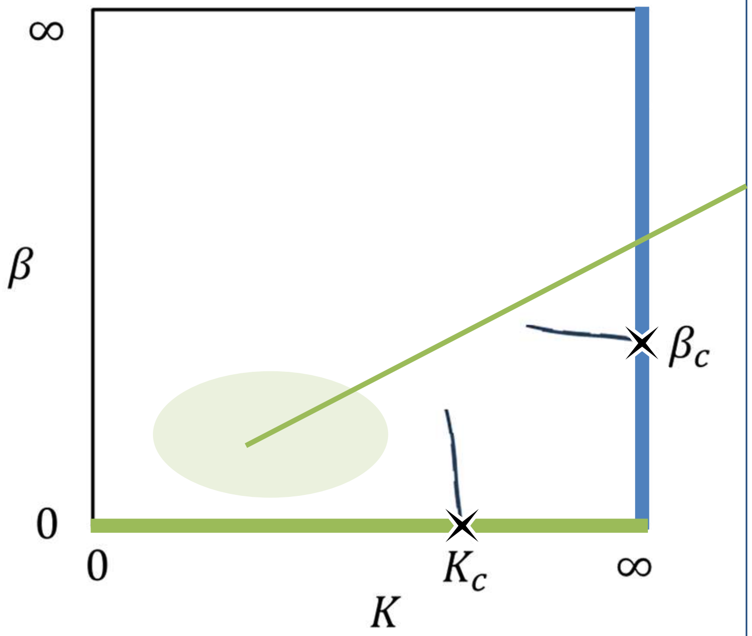
$\longrightarrow$ Gauge field is **higgsed** and becomes **massive**

Perimeter law of Wilson loop:

$$W_\Gamma \sim e^{-\mathrm{Perimeter}(\Gamma)}$$

Other regions (Confinement phase)

The conjectured phase diagram:

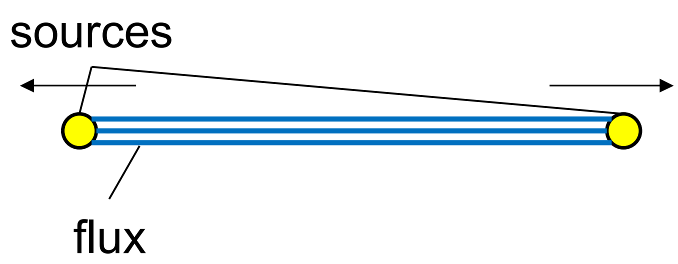


Confinement phase

Area law (?) of Wilson loop:

$$W_\Gamma \sim e^{-\mathrm{Area}(\Gamma)}$$

$\longrightarrow$ Flux tube is squeezed **(massive gauge field)**



Other regions (Confinement phase)

$$S[U_\mu(\boldsymbol{r}),\phi(\boldsymbol{r})] = \frac{K}{2}\sum_{(\boldsymbol{r},\mu,\nu)} \mathrm{Tr}[\square_{\mu\nu}(\boldsymbol{r}) + \mathrm{h.c.}] + \frac{\beta}{2}\sum_{(\boldsymbol{r},\mu)} [\phi(\boldsymbol{r})D(U_\mu(\boldsymbol{r}))\phi^\dagger(\boldsymbol{r}+\boldsymbol{e}_\mu) + \mathrm{c.c.}]$$

The conjectured phase diagram:

Confinement phase

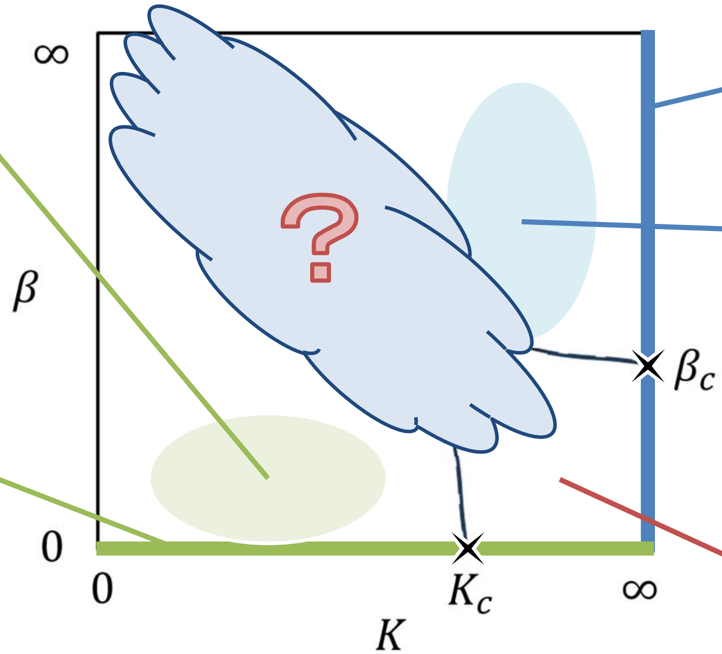
→ **Area law (?)**

Considering the effect of screening, we can have **Perimeter law** here

Pure gauge theory

→ **Area law**

$$W_\Gamma \sim e^{-\mathrm{Area}(\Gamma)}$$



Ising model

Higgs phase

→ **Perimeter law**

$$W_\Gamma \sim e^{-\mathrm{Perimeter}(\Gamma)}$$

Coulomb phase

→ **Perimeter law**

Screening in confinement phase

In confinement phase ($K \ll K_c,\ \beta \ll \beta_c$), matters can **screen** sources.

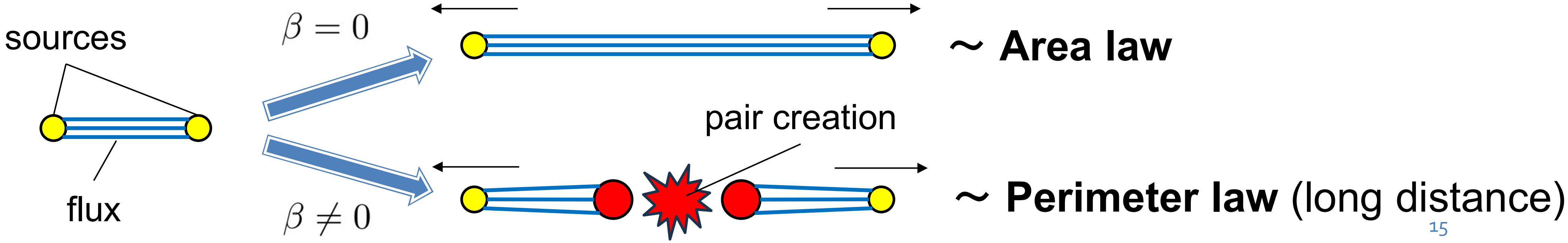
For $\mathbb{Z}_2$ gauge theory:

$\beta = 0$

$$W_\Gamma \sim (\tanh K)^{RT} + \ldots \sim e^{-\tau \cdot \mathrm{Area}(\Gamma)}$$

$\beta \neq 0$ **(screened)**

$$\begin{aligned} W_\Gamma &\sim (\tanh \beta)^{2(R+T)} + \ldots + (\tanh K)^{RT} + \ldots \\ &\longrightarrow e^{-\lambda \cdot \mathrm{Perimeter}(\Gamma)} \end{aligned}$$



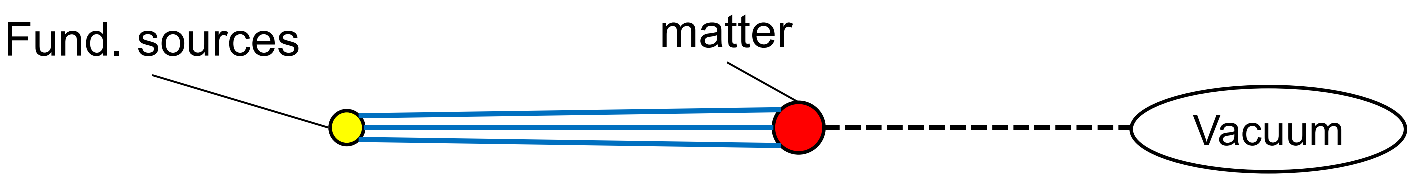
Screening in confinement phase

In confinement phase ($K \ll K_c,\ \beta \ll \beta_c$), matters can **screen** sources.

For G gauge theory:

$\beta \neq 0$ **(screened)**

If Higgs belongs to the rep. R such that the fusion satisfies



$$\overline{\mathrm{Fund.}} \otimes R = \mathrm{id.} \oplus \cdots$$

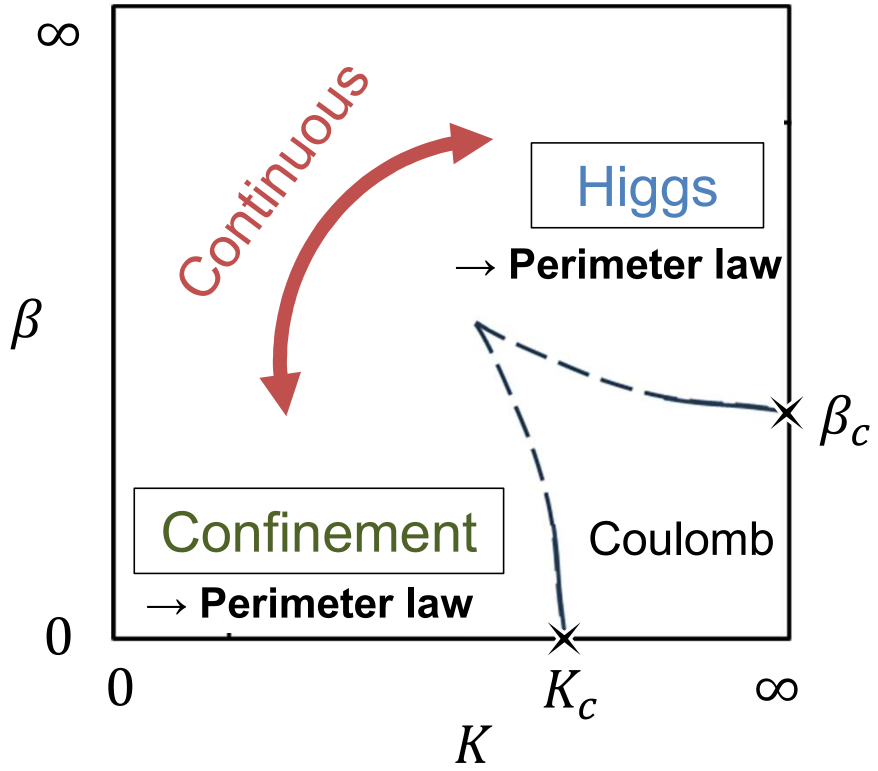
then flux **can be screened** via pair creation $\Rightarrow$ $W_\Gamma \sim e^{-\lambda \cdot \mathrm{Perimeter}(\Gamma)}$

$\Downarrow$

If screened, **confinement phase and Higgs phase are connected**

The whole phase diagram

The whole phase diagram is given by the following:



If fundamental matter exists,

1. Confinement phase

→ sources are screened (~perimeter law)

$$\overline{\text{Fund.}} \otimes \text{Fund.} = \text{id.} \oplus \cdots$$

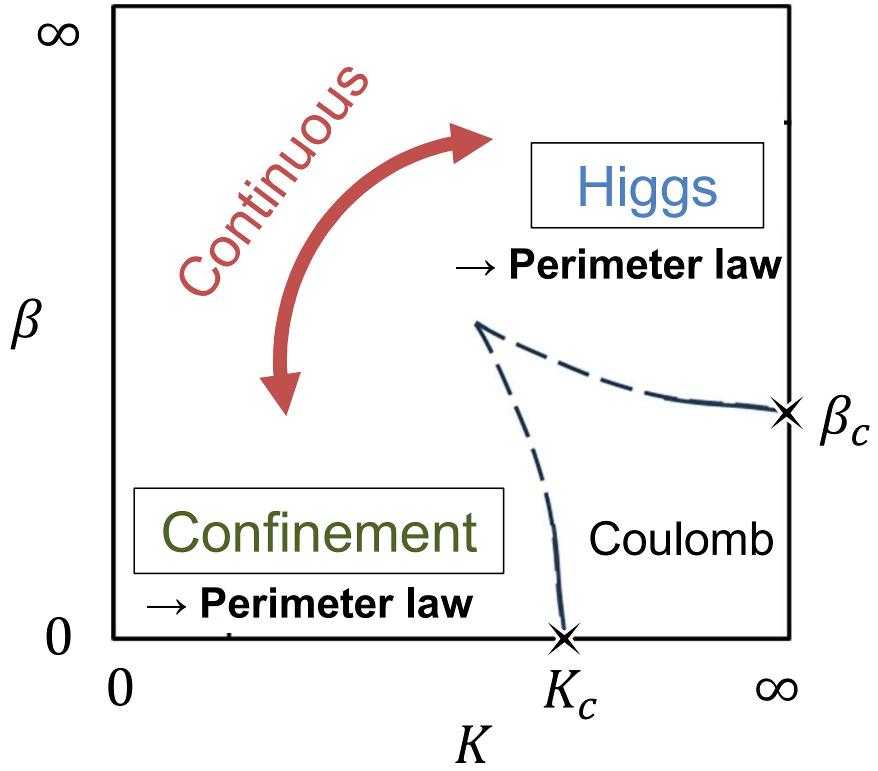
2. Higgs phase

→ G is completely broken

$$G \to 1$$

The whole phase diagram

The whole phase diagram is given by the following:



The gauge-invariant operator

$$\phi(0)\left[\prod_{(\boldsymbol{r},\mu)\in\Gamma} U_\mu(\boldsymbol{r})\right]\phi^\dagger(\boldsymbol{r})$$

connects the two states continuously:

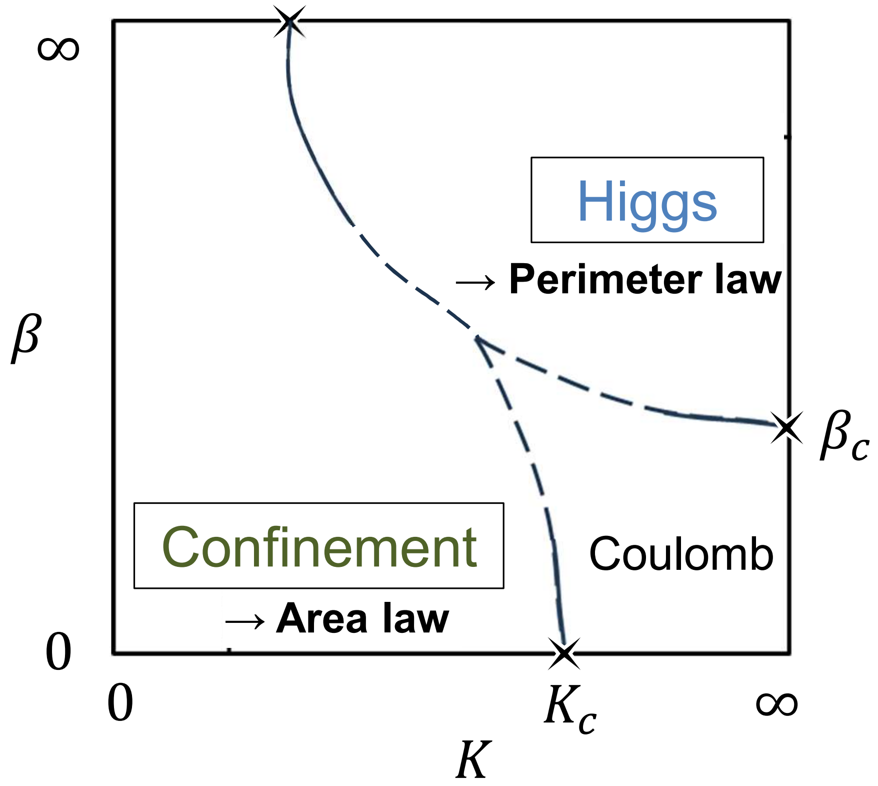
1. a **“meson”-like particle** in conf. phase
2. a **massive gauge boson** in Higgs phase

⇒ Phases are **connected continuously**
(Fradkin-Shenker continuity)

The whole phase diagram

e.g.
- Adj. in $SU(N)$
- charge q in $\mathrm{U}(1)$

The whole phase diagram is given by the following:



If higher rep. matter exists,

1. Confinement phase

→ sources are not screened (~area law)

$$\overline{\text{Fund.}} \otimes R = \cancel{\text{id.}} \oplus \cdots$$

2. Higgs phase

→ G is partially broken $G \to H \ (\subset G)$
and Higgs takes a value in G/H

Phases are **separated by a line**

Conclusion

If Higgs obeys…

Fundamental rep. (SSB: $G \to 1$)

$\longrightarrow$ **No boundary** between Higgs and confinement

	Gauge symmetry	Massless gauge boson	Spectrum
Confinement Phase	Broken	No	Only G-singlet
Higgs Phase	Broken	No (If $K \gg K_c$, massive)	Only G-singlet
Coulomb Phase	Unbroken	Yes	Charged state

Other higher irrep. (SSB: $G \to H(\subset G)$)

$\longrightarrow$ Boundary can exist